Scuola universitaria professionale della Svizzera italiana
Dipartimento formazione e apprendimento

SUPSI

LAVORO DI DIPLOMA DI

MASSIMO BANFI

MASTER OF ARTS IN SECONDARY EDUCATION

ANNO ACCADEMICO 2014/2015

BEYOND THE HORIZON: A NON-EUCLIDEAN RIEMANNIAN LABORATORY IN THE LOWER MIDDLE SCHOOL

RELATORE

DR. ALBERTO PIATTI

To my dad that opened his child's imagination with the curvature of Space.

I am very grateful to Dr. A. Piatti for having led me to structure and complete a potentially unending work.

To the DFA (Dipartimento Formazione Apprendimento), in particular to his lecturers and teachers, I owe the entire philosophy of grounding mathematics and its learning into the physical world and the real student – without which my epistemological approach would have been unthinkable. The in-service-teacher, Giancarlo Sonzogni, and the Scuola Media Gravesano, were very helpful and provided me with perfect working conditions to implement the laboratory in one of their classes. The professors, E. Franchini and A. Gambini of the Università di Bologna, kindly found and delivered to me 10 complete Lénárt kits on a very short notice – together with precious recommendations about the practical use of the kit. Professor P. Morando, of the Università Statale di Milano, generously gave me her feedback and support on the many subtle mathematical challenges of general relativity and differentiable manifolds. Dr. A. Kennington, from Melbourne, graciously assisted me with some foundational aspects of differential topology.

To the students that shared with me their 22 different "windows on the non Euclidean world" – each one enriched by that humanity that transcends any science.

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SUPSI

Abstract

Banfi Massimo

Master of Arts SUPSI in Insegnamento per il Livello Secondario 1

Beyond the Horizon: a Non-Euclidean Riemannian Laboratory in the Lower Middle School

Alberto Piatti

Non Euclidean geometries are by act of birth abstract and axiomatic, though they have been made popular by their 2D models – surfaces of sphere and hyperboloid, by which they are commonly proposed at our school level. We questioned the effectiveness of these approaches for learning non Euclidean geometry in our lower middle school and designed and implemented a laboratory based on an radically different non Euclidean epistemology that we developed out of a focused analysis of the historical epistemological processes that led from Euclid's geometry to Riemannian manifolds and to general relativity. In conjunction, we critically revised our psycho-educational frame of reference for geometry cognition and learning. In the resulting laboratory, the students started by building outdoor an embodied sensorimotorial model of going straight (straight line) - discerning curvature of the ground (manifold) and "curvature" of the movements. On their "flat land" discovered non Euclidean behaviors of their straight lines, circumferences and triangles – when extended beyond the horizon. Initially students were puzzled by the replacement of static geometry with kinematics and by the paradoxical non Euclidean behaviors of the figures. Faster than expected, many of them, considered "evident" the non Euclidean behaviors and were able to effectively use curvature as a means to explain them. Many students appeared having reorganized their conceptions of straight line, curvature as well as convictions about triangles or circumferences, whereas "the parallels "were epistemologically and didactically more difficult.

Index of contents

Chapter 1: The research

1.1 What this work is about

This study concerns the problem of effective learning in situations of non Euclidean geometry, the solution we designed and implemented – based on the epistemology of Riemannian manifolds and of relativistic kinematics[1], and the students response to it.

1.2 Our environmental conditions

In the Swiss Italian middle school there are four grades – the lower, first and second, and the higher, third and fourth. Our class is a second grade of 22 students – males and females in equal number, aged between 11/12 and 13. The laboratory was granted 15 hours in a period of three months to be taken out of the normal hours of the in-service teacher. General cognitive and mathematical competencies presented large variations. Indeed, in the lower middle school, differently from the higher, there is no curricular differentiation for mathematics, in line with the principle of "Inclusiveness". The class had no previous knowledge of the metric aspects of the circumference and of the sphere. At the elementary school, students had learnt about circles and spheres as 3D shapes (i.e., proto-topology) and, in the middle school first grade, about the metrical cube.

The curricular plan for the lower middle school[2] privileges a student centered teaching/ learning of socio-constructivist inspiration and based on laboratories and the theory of the situations of Brousseau. For geometry in particular, the main approach is that of integrating direct perceptive experiences of "reality and its objects" into an "empirical geometry" including some degree of abstraction and formalization (see, for example: Arrigo & Sbaragli, 2004; Sbaragli & Mammarella, 2010 and Cottino et al., 2011) - though never at the level found in the Euclid's books.

The pre-service teacher (in the following, the teacher) that prepared this work had no previous knowledge of the students and initially leveraged on the in-service teacher knowledge.

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[1]A physics of movements respecting general relativity equivalence principle.

[2] Piano di formazione della scuola media (2004).

1.3 The topic: non Euclidean geometry

Geometry commonly taught in lower middle school concerns the contents (as opposed to the "structure/method") of Euclid's geometry. These contents are meaningful to students, for they correspond well to the geometrical experience students have of their world.

Euclid structured the contents of his geometry by a hypothetical-deductive system consisting of some initially given definitions, five postulates and some common notions, from which the theorems are inferred by precise rules of logic – as we argue in the paragraph, Euclid's geometry as a system, of chapter 3.

The system is founded in distinctive features of the Greek gnoseology and science – as we discuss it in the paragraph, Euclid's geometry deep foundations, of chapter 3 - that were revamped, starting in XVII century, by the scientific revolution in Europe, confirming and reinforcing the validity of Euclid's geometry contents. For example: Newton universal gravitation theory is founded on Euclidean geometry of absolute space and time and it is structured as a hypothetical-deductive system mimicking Euclid's geometry (see Newton, 1999).

In the system of Euclid, there is one postulate, the fifth, that is about "the uniqueness of the straight line parallel to a given straight line through a given point" – explained in the paragraph, Euclid's geometry five postulates, in chapter 3. For more than 2000 years many tried unsuccessfully to derive it from the other postulates. No one thought of it as unnecessary for geometry, because, as we argue in chapter 3, geometry was about "true knowledge of the world" and not about studying formal systems or speculative-geometries. It is with Gauss, Bolyai, Lobachevsky and their works that the "physical world" became irrelevant to geometry and that geometry became accountable to logic only – as we explain in the paragraph, Non Euclidean geometries as per their origins, in chapter 3. Non Euclidean geometry was born: the V postulate was rejected, and the consequent geometry – today known as hyperbolic, explored and, after many years of further works, proven and accepted as mathematically meaningful. Other non Euclidean geometries followed - by "removal" of other postulates or base elements.

1.4 A promising topic

As effect of these removals, straight and curved lines, triangles and their angles, as well as the relationships between circumferences and their diameters, would have strikingly paradoxical behaviors challenging common wisdom. In particular, a most unquestionable common sense fact - that the internal angles of a triangle add up to a flat angle, simply ceases to be valid: the sum may

take any value, depending on the specific triangle. Similarly, the ratio between the length of a circumference and its diameter - that students are taught to be invariable and equal to π, may take any value as well.

If we take the example of the particular non Euclidean geometry known as "spherical", the internal angles of a triangle always add up to more than one flat angle and the ratio is always less than π – whereas, for the "hyperbolic" geometry, it is just the opposite: the sum is less than one flat angle and the ratio is more than π.

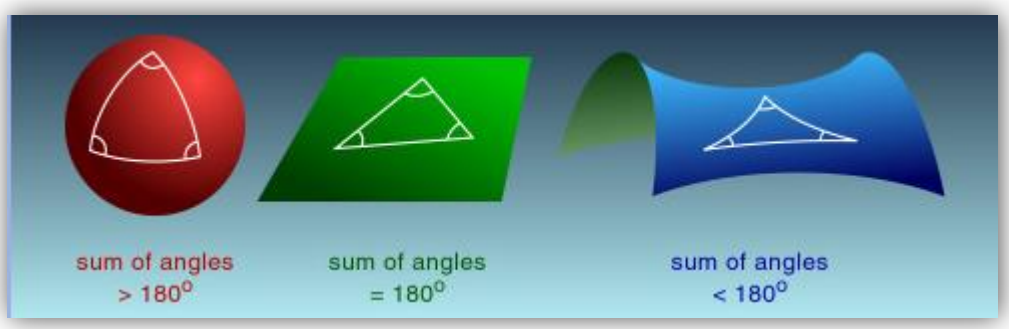

Figure 1. 1

Non Euclidean geometries have impacted and inspired both philosophy (e.g., Bachelard) and artists (e.g., Picasso, Braque, Boccioni, Dalì), who could find in their ideas and history - and in the ideas of multidimensional space and space time, powerful metaphors or means to break free from supposedly universal laws of space or paradigms.

Finally, the history of the discovery, development and application of the ideas of non Euclidean geometry, is constellated by men showing faith and devotion to their ideals, determination and belief in the possibility and progress of true knowledge and in its being a man's worth mission: Lobachevsky, Bolyai, Riemann, Clifford, Helmholtz, Poincaré, Einstein – to name a few.

At a first sight, therefore, a laboratory on non Euclidean geometry appears as having a high potential for broadening and deepening students cognition and learning of geometry - as well as for providing them with examples of how critical enquiry may open up new perspectives on apparently unquestionable received truths (the proverbial "mathematical" ones).

To a teacher, it appears offering opportunities for a deeper understanding of students' cognition and learning of geometry as well as for developing his mathematical transposition skills on a topic for which, in our environmental conditions, there are no standard references.

1.5 A problematic topic

Yet, since we assume that what is relevant to our students is the "world of their experience" – and not abstract theories that rest on their mathematical validity only, we have some issues.

Indeed, it is not clear to us, how could non Euclidean geometries possibly challenge our students believes, or any common truth or wisdom, when the very changes that they make to the Euclidean geometry (the changes to one or several of the original Euclid's postulates) make geometry contrary to or irrelevant for how students experience the world. Indeed, these geometries statements are simply not true and inapt to represent our experience of the behaviors of triangles, parallels and circumferences or, if we assume that the non Euclidean deviations are under our perceptions or empirical threshold, they are simply irrelevant to our students. In either case, we find it difficult to argue about their having any bearing in modifying any student's paradigms, conceptions, schemas, a priori cognitive structures, etc. regarding the geometry of the world of his experience. In other words: (non Euclidean geometry) by making the world irrelevant to mathematics, mathematics become irrelevant to our students.

Instead, what one would need, is a situation in which students may experience the "non Euclidean" as a real, authentic aspect of their world.

These perplexities are confirmed by the strategy commonly followed by laboratories on non Euclidean geometries: proposing an intrinsic (i.e., "seen from within") geometry of some surface, most commonly the spherical surface - on the basis that these surfaces are "models" of the abstract non Euclidean geometries. The figure below shows common pictures for the spherical, the hyperbolic and the flat surfaces (the last would correspond to "Euclidean geometry").

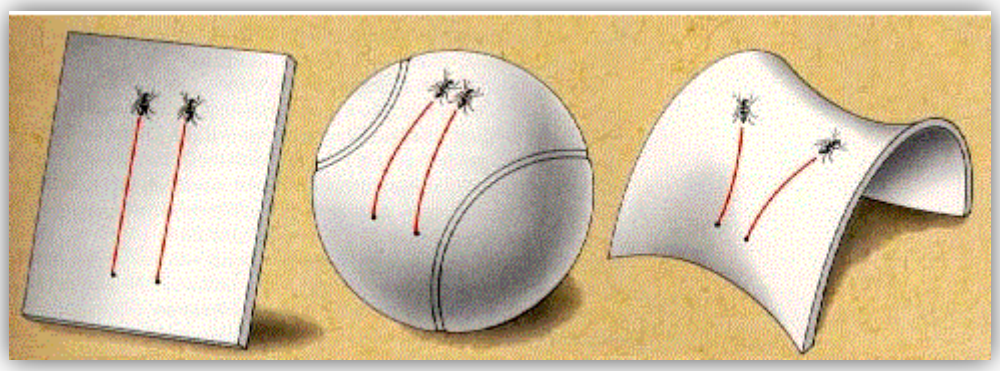

Figure 1. 2

In our opinion, that is not how our students experience these curved surfaces: they experience them from the outside world - embedded in a three dimensional Euclidean space, resulting in a learning

that is not authentic and having no bearing on their conception and perception of geometry of their world.

Although these issues are critical for our environment, we have found them, neither clearly pinpointed, nor resolved in any research or work known to us. Wondering why, we have made the conjecture that other laboratories might

-> start from the body of knowledge of non Euclidean geometries as a "given"

and from there, didactically transpose it (Chevallard, 1985) in their best possible ways with respect to their students profiles.

In addition, perhaps, people who are very accustomed to have mathematics as part of their life, may assume that "mathematically valid" may imply some degree of "meaningfulness". Although we agree that this may be the case for students who "experience or attribute a certain degree of reality to the world of mathematical images and models[3]" - as it could be the case for many of the mathematicians[4] or some students, we are convinced that this is not the case for the major part of our students.

Whatever the reasons, in line with the principles of our lower middle school curricular plan,

-> we start from the student,

i.e., from what scientific literature tells us about the necessary conditions for an effective learning of geometry with respect to non Euclidean aspects and by this critical-grid we look at the body of knowledge.

Metaphorically speaking: "It is to the body of knowledge to prove itself teachable via an effective transposition – not to the students to prove that they can learn a "falling from above" body of knowledge". Indeed, by taking such a neat stance, we were led to change (the epistemology of) the common body of knowledge of non Euclidean geometries.

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[3] Or, paradoxically, for students that learnt geometry as a "school thing" – disjoint and with no bearing on the world of their experience.

[4] In the words of Bolyai, one of the discoverers of the non Euclidean geometries, "I have discovered things so wonderful that I was astounded ... out of nothing I have created a strange new world" – letter to his father in the year 1823 (source: On Line Math Tutor University of St. Andrews, Scotland). The world of which he was writing was a purely formal axiomatic geometry that had yet to be proved mathematically meaningful and had no concrete model in our physical world – and yet he was writing of it as if it were "real".

1.6 An opportunity

Despite the serious problematic aspects hinted above and the insistence on a "priority" of the student over the body of knowledge, we are convinced that there are good reasons to take the challenge. In a different form, the "non Euclidean" is well real: in the "big picture of general relativity" the world is, as we will argue in chapter 3 and eventually in paragraph 3.9, non Euclidean indeed[5]: space time is curved and straight lines, triangles and circumferences display non Euclidean characteristics. Since Einstein built the big picture on top of still another geometry, the Riemannian manifolds, we have thought to overcome the apparent irrelevance and meaningless (for our students) of original and traditional non Euclidean geometries by rethinking "non Euclidean" in terms of Riemannian manifolds and by rethinking geometry in terms of physics of movements[6] (of general relativity). We note that a conception of mathematics as (mathematical) physics is advocated, on a much broader scope, also by some prominent mathematicians, notably, Arnold (see paragraph 3.8).

1.7 The research hypothesis

Our hypothesis is that:

1. by putting the student in the center and carefully lining out the essential psycho-educational requirements (necessary conditions – of the kind we mentioned in the previous paragraph) most critical for an effective learning of (non Euclidean) geometry in our environment;

2. by a strictly focused analysis of the epistemology of Riemannian manifolds and general relativity;

it is possible:

3. to articulate a specific epistemology of geometry (reconciling Euclidean and non Euclidean) allowing a didactical transposition in a laboratory fulfilling our psychological and educational requirements above, in which the students' understanding of the concepts of straight line, triangle, circumference, as well as of their properties, will be broadened and deepened by working on

[5] This is all but a common and universal view. As we will argue in the paragraph 3.9, Is our Universe Euclidean?,, the world is indeed non Euclidean in a precise sense, but the common "many geometries paradigm", which originated with the divorce of mathematical geometry from physical geometry, is obscuring the discussions.

[6] Generally, we avoid saying "kinematics", because may imply that it is the study of movements without concern for fields (force) – that are instead essential to introduce non Euclidean effects.

paradoxical situations (non Euclidean) appearing on their land on large scale, e.g., lines both curved and straight, circumferences that are straight, and intersecting straight parallel lines;

as a result:

4. the students will extend, deepen and reorganize their conception of geometry from “a mathematics of static figures” to a “(mathematical) physics of movement”, “accounting for both Euclidean and non Euclidian behaviors”.

1.8 The research questions

Based on the previous discussion and hypothesis, our research questions are:

1. What is a set of essential psycho-educational requirements that are most critical for an effective learning of (non Euclidean) geometry in our environment?

2. What is a specific epistemology of geometry (Euclidean & non Euclidean), and a didactical transposition, built on the key ideas of Riemannian manifolds and general relativity, that can fulfill the psycho-educational requirements?

3. What is a laboratory’s design based on the above epistemology and transposition and driven by the above requirements?

4. What is the response of the students, in broad terms of cognition and meaning, to the implementation of such a laboratory?

Chapter 2: Our essential requirements for effective learning

Objective of this chapter is to answer the first research question:

1. What is a set of essential psycho-educational requirements that are most critical for an effective learning of (non Euclidean) geometry in our environment?

Our requirements have a pedagogical priority and provide focus to the work on epistemology. For this reason their chapter precedes the chapter of the epistemological analysis. Yet, actually, we proceeded in "feedback": psycho-educational requirements bent our epistemological research, which in turn, restructured the requirements[7]. Because of this, despite our editing effort, it might be necessary to have read some parts of the epistemological analysis to completely understand the requirements.

Our requirements are, evidently, a matter of subjective choice – which we endeavored to make it as rational as possible. They result from a number of decisions, including: who are the authors and theories of reference; what aspects of these theories to consider and how to interpret them; how to texture together these interpretations into consistent guidelines for our context and topic; what to consider implicit and commodity and what critical and compulsory to be made explicit; what we think makes sense for our students, of our school, of our region, of our time, etc. and what we think does not make sense to explore. And, as mentioned, our requirements analysis is influenced by our knowledge of modern mathematical physics models and thinking.

Our approach will be naïve – in a sense similar to that implied by naïve in "naïve set theory": a theory of sets founded in natural language, instead of formal logic. For example, we take the theory of didactical situations in the sense of Brousseau (1997) as our reference, but without being concerned with the issue of its grounding in psychology, for example, as pointed out by Crahay (2013). Similarly, we assume key elements of cognition in psychological terms (for example, according to Piaget (1967), or across different theories as in Miller (2011) and in Crahay (2011)) as well as in neurophysiological terms (for example, according to Teissier (2006)) without being

[7] As is happens when a situation lent itself to be modelled as a dynamical system - our « topic » example being: "Space tells matter how to move" "Matter tells space how to curve" (Misner et al. (1973), p. 5). See also more below in this chapter our discussion on dynamical system.

concerned with how they correspond to each other and to general information processing theory and niche psychologies – especially, dynamical systems based (Miller, 2011).

Even for our basic assumption about "the nature of mathematical ideas from a cognitive perspective", that is: "[…] our ideas are shaped by our bodily experiences - not in any simpleminded one-to-one way but indirectly, through the grounding of our entire conceptual system in everyday life." (Lakoff & Núñez, 2000, p. XIV) - we will not examine the presuppositions of such a theory or to which degree would it be consistent with other perspectives – as those mentioned above.

2.1 Analysis

Scientists of different fields agree that psychological cognition of geometry is rooted in a personal, body-centered and direct sensorimotorial experience.

Indeed, in his epistemology of geometry, Poincaré, a scientist and one of the major geometers of the XX century, expressed clearly that at the root of space there is movement. Arzarello (2012) underlines that "Per Poincaré è la presenza dei corpi, del nostro corpo in particolare, e dei movimenti, dei nostri movimenti, a generare la nozione di spazio" (p. 3).

In his studies of mathematical education, Freudenthal, a mathematician and prominent figure in mathematical education, makes clear that geometry, before becoming abstraction is about living personally objects and movements: "Geometry is grasping space...that space in which the child lives, breathes and moves. The space that the child must learn to know, explore, conquer, in order to live, breathe and move better in it" (Freudenthal, 1989, p. 48).

In his genetic epistemology, Piaget, a key figure of developmental psychology and cognition, argues that forms and space of physical reality are built directly from perceptions at the sensorimotorial level, are integrated in representations and images first in the operational stage and later in higher stages (Piaget & Inhelder, 1967). The learning (development) is egocentric, that is, it is lived personally and from the perspective centered on the person's body.

Piaget discussed at great length the "representational space" – a cognitive higher reprocessing of "perceptual space" (the space of raw perceptions built in the early infancy), in Piaget & Inhelder, 1967. Two aspects of it are key to us:

1. the forms of this space are built by direct actions of the children with the world – not by passive observation

2. in the early stages representational space is only “topological” – there is no concern about length, area, volume, angles, etc., but only shape (homeomorphisms). Only, in later stages a metric structure is added

The key implication for us is that, because non Euclidean situations challenge the very roots of geometric cognition, our laboratory will have to start from sensorimotorial images and the early tiers of representational space, non metric geometry: with students acting in the space of their world.

In their educational well known theory of geometry learning, the van Hiele’s, argued that children pass through five levels of “geometric cognition” (Van Hiele, 1986). Today, the levels are used, numbered and referred in different ways – depending on the author, the field, etc. (Mason, 2002), but the common list is: visualization, analysis, informal deduction, deduction and rigor. The first two levels are more critical in our work, for the transition from the first to the second corresponds to the shift from a holistic cognition of geometry by shape (in our case, for example, recognize a triangle as such because it “looks like” the prototypical idea we have of the triangle), to a descriptive and analytical understanding (for example, when we are forced to analyze “what is it that makes a triangle a triangle?” to eventually decide whether “a spherical biangle is or is not a triangle”). To a minor extent, informal deduction will appear critical as well as in our laboratory, because of some didactic device we chose to apply, namely, “empathic quiz’s” in which the students have to apply heuristic and deductive reasoning.

-> A gradually driven didactic progression from level 1 to level 2 appears to be beneficial for our laboratory topic – where we introduce entirely new concepts as “curvature of the underlying”.

We shall note that the van Hiele’s model has been changing, is used in different ways and, as Piaget’s, allows for having children that are at the same time in different levels for different topics, for regressions, etc. (see for FAQ about van Hiele’s model in Mason, 2002).

The van Hiele’s model and the staged process of Piaget are different in several respects. One that interests us, is the degree of correspondence with modern mathematical structures. Piaget sees children cognitive development, we might say, mirroring to a good degree the mathematical structures from simple to complex. The child walks from topology, to projective, to affine geometry, etc. (Freudenthal, 2002, p. 27 and 28). It is clear that Riemannian manifolds would have no use for our students – if the genetic epistemology had to follow such a conception. We share the

broad view expressed in Freudenthal (2002) that the conceptual picture of mathematics and geometry that is apparent in Piaget's work corresponds well to a structuralist or Bourbakist picture.

Indeed, it is at this point, that we can no longer adhere to Piaget's work. The conception (epistemology) of geometry that we have taken for this work [8] is different: geometry as (mathematical) physics of movements. Anticipating some of the epistemological analysis, Einstein's work may be interpreted as a revolution in such a sense of geometry (see the corresponding paragraph in chapter 3) – a revolution that started already with Riemann's insights in physical curved space. The root statements of Poincaré, Piaget and Freudenthal are about physical experiences at a "level" where geometry has yet to divorce from physics of movements. In so far they are good to us, for we can develop from a conception of geometry as (mathematical) physics of movement. Not so when Piaget articulates its genetic epistemology around mathematical-structures of modern mathematics – as said above. As proven by our laboratory, Riemannian manifolds can be made into lively experiential geometry (physics of movements) for our students.

->The key implications for us, is to "stick on the original sensorimotorial conceptions", by developing a geometry as physics of movements.

With this mindset, the central building block of geometry is the movement. And of all possible movements, the straight movements or straight lines, are to be (re)built in the deepest concrete level, the sensorimotorial and kinematically. The straight line of traditional geometry, idealized entity, is expelled – as it is from general relativity: "The "ideal straight line" is a myth. It never happened, and it never will" (Misner et al., 1973, p.19) - commenting about the old idea that there is an absolute and perfect space to ground such an idea, which is an idea that is at the base of any straight line being designed in our school geometry.

-> Some studies of neurophysiology and geometric cognition propose a conception of geometry very similar to "ours" (physics of movements) – in contrast to that of Piaget (mathematical geometries).

In them, we find also a model of the straight line that goes in the direction we need in this work; In Longo & Viarouge (2010):

[8] We did choose this conception not out of esthetical or a priori rationales, but only because it is instrumental to resolving our overall problem with student's learning.

"[…]a gestalt which is constitutive of Mathematics: the widthless line. From the cognitive standpoint, it is possible to refer first to the conceptually simultaneous role of:

- The saccadic eye movements preceding the prey, tracing its continuous path,

- The "vestibular line" (the inertial stability of the body, gided by the vestibular system, which contributes to the memorization and continuation of inertial movement),

- The visual line (which includes the direction detected and anticipated by the primary cortex,[…]. […] we do not understand what a line is […] without the seen gesture, or even without it being drawn on a blackboard, or felt, appreciated by the body, in the evocation made by the gesture of teacher." (p.13)[9].

-> The straight line model, sensorimotorial and body centered, to be proposed to our students as a concrete entity of a geometry that becomes physics of movements, finds correspondence also with some neurophysiology basis[10]

Yet, as we will discuss in the epistemological analysis of the straight line model in chapter 4, these neurophysiological studies assumed physics of movements that is Newtonian - in the sense that: a straight movement is a movement "free from acceleration". On the contrary, the conception of physics that we were led to take is that of general relativity: a straight movement is a movement "freely falling in the gravitation field". As it will be explained, this is the only way (we could find) to "encode" non Euclidean geometry inside our "local physics (geometry)". This model requires a slightly but critically different "vestibular" model of the straight line - a model valid for both Euclidean and non Euclidean straight lines.

The issue we have found above with Piaget and the one with the neurophysiology researches are examples of issues that we have found with other specialist theories as well. In our opinion, the rationale is obvious. Many theories that have developed and articulated the initial basic statements of Poincaré, Piaget and Freudenthal, without a background of a specific conception of geometry as physics of movements and physics as general relativity (essentially: equivalence principle and curvature of space time manifold)[11],

[9] A more complex and extensive treatment is in Teissier, 2006.
[10] Though "imperfect" as we clarify more below.
[11] That to us are only means to resolve our problem of finding non Euclidean experiences in students Euclidean experience of the world.

-> have carried in themselves a number of aspects, or paradigms, specific of a conception of geometry as static mathematics, as Euclidean, or of a conception of physics as Newtonian, that make them difficult to be applicable to our situation.

As a further example, we very shortly discuss some of the obstacles we met with one of the embodiment cognition theories that is more developed and provides a systematic framework for mathematical learning in the "body centric" approach we also use.

In very broad terms, the embodiment thesis (Wilson & Foglia, 2011):

"Embodiment Thesis: Many features of cognition are embodied in that they are deeply dependent upon characteristics of the physical body of an agent, such that the agent's beyond-the-brain body plays a significant causal role, or a physically constitutive role, in that agent's cognitive processing."(p. 12 and 13) - is consistent with the initial positions of Poincaré, Piaget and Freudenthal – and our requirements.

Yet, when this general statement is developed into a theory for mathematics learning, the underlying "assumed conception of geometry" becomes, to us, an obstacle – as we clarify with a concrete case. Consider the following schema proposed by Tall (Tall, 2013, p. 151). The "leit motif" of a geometry rooted in our physical experiences of the world, and concerning the world is evident – and we find ourselves aligned to it.

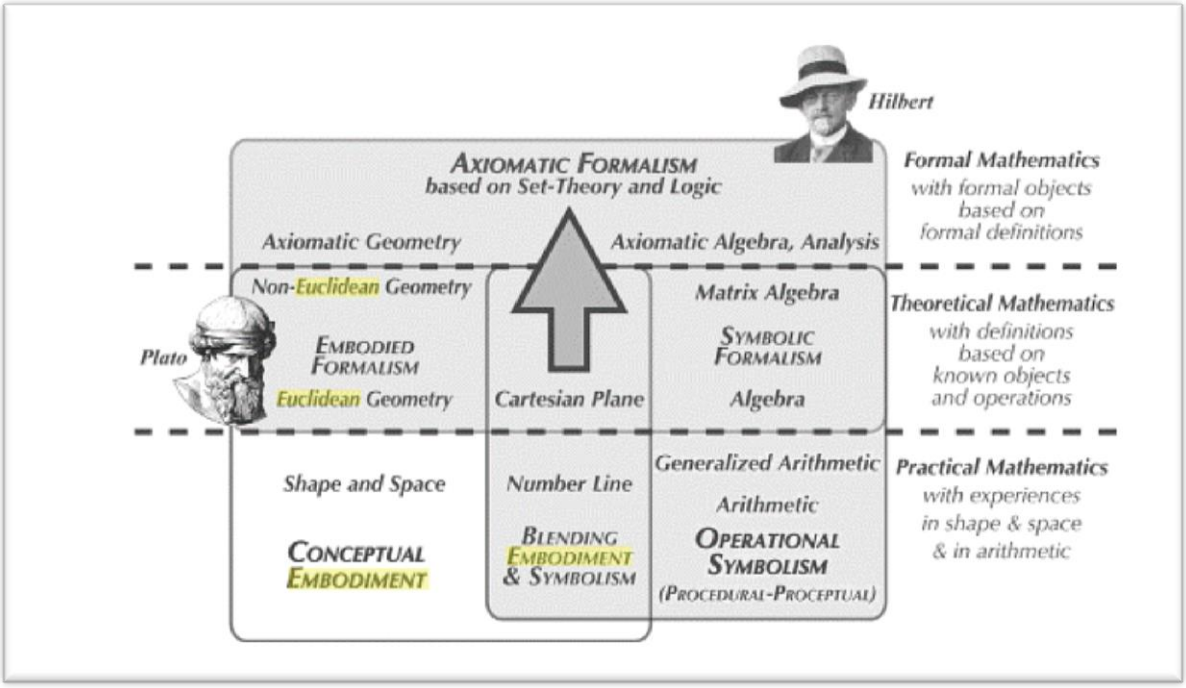

Figure from, Tall, D. (2013). *How Humans Learn to Think Mathematically Exploring the Three Worlds of Mathematics.* Cambridge University Press

When we look at the left side, the only one that concerns us, geometry, we recognize that it is built from embodiment, and for the physical world, and it is systematized by going up to theoretical mathematics.

Yet, because of our conception of geometry/ mathematics, to us,

-> it is not geometry/mathematics that is embodied and that ascends to axiomatic geometry

-> it is physics of movements that is embodied and that ascends to axiomatic gauge theories.

From our standpoint, the embodiment theory of Tall looks like "as if physics of movements had been dismembered to extract from it embodied geometry".

This difference impacts directly on the foundations of the theory presented by Tall and makes it difficult to reuse it for our purposes. As we said, this is a general circumstance. Indeed, we know of no theories built on a conception of geometry as ours, usable for our aims and for our environment.

Even if we left aside physics of movements, theories known to us assume a conception of Euclidean and non Euclidean geometries at odds with the one we aim to implement in our laboratory – so creating another obstacle for reuse.

Indeed, if we go back to Tall's schema, we see that in it non Euclidean geometries sit farther from embodiment than Euclidean geometry. Instead, because of our aims, to us, Euclidean geometry would not be less formal and abstract than non Euclidean geometries (!). Indeed, in our opinion, an embodied cognition of Euclidean geometry is a priori impossible, for it requires an unbounded neighborhood of perception[12] – which is contrary to the definition of human agent. What we have, instead, is an embodied cognition of a geometry that is "locally Euclidean". But any Riemannian geometry is also locally Euclidean. Therefore, our embodied geometry is more correctly defined as Riemannian (a particular case of which is Euclidean geometry), not Euclidean and not non Euclidean. We anticipate that it is because of this crucial point that we can lead our students into a non Euclidean extension of their local geometry: by the mediation of artifacts of different types, they can discover the non Euclidean situations of large scale neighborhoods on the spherical surface of the Earth. Therefore, if we were to adapt Tall's schema to our problem, we would have to put at the embodiment level "Riemannian geometry" and leave for the higher level Euclidean, non Euclidean, etc. Actually, as said above, we would also have to change "mathematics" to

[12] One has to check the parallels postulate or any equivalent – e.g., sum of internal angles of triangles of unbounded scale.

mathematical physics of movements (relativistic) and Riemannian geometry in Riemannian kinematics.

But, this would only be the beginning. Geometry as physics of movements brings up a number of other fundamental concerns, alien to Tall's scope. One example, also of great historico-epistemological relevance, may be brought in by the dispute in which Galileo found himself entangled: "[…] the Earth moves around the sun rather than the sun around the earth […]" (Noll, 2004, p. 6) – on which, our modern scientific view is: "[…]neither of these assertions makes any sense because frames of reference are not specified." (Noll, 2004, p. 6).

Indeed, the above two models have no sense per se. They are relative terms: depend on a system of reference[13].

One may appreciate that this is a fundamental cognition node for our cognitive embodiment of geometry (as physics of movements) that cannot be "dis-embodied" – in our situations and epistemology (whereas it is not embodied in Tall's model – for the mathematics of his model is disembodied from physics).

Indeed, regardless of one's dealing or not with it, this "node", is present in the student kinesthetic perceptions. Even at the neurophysiological level, frames of reference appear to be deeply embedded into neuro-cognition and inextricably connected to the illusion of a "physical space" – as it was explained by Pinker (1997), cited in Noll, 2004, p.7[14], and as it has been studied and further developed in recent researches (for example, Klatzy, 2012, in particular the egocentric and allocentric systems in *The embodied actor in multiple frame of reference*, R. L. Klatzy and B. Wu).

To summarize: the conception of geometry we adopted is that of a physics of movements, in which straight movements are not acceleration free, but freely falling (general relativity equivalence principle). Any psycho-educational analysis of geometry learning assumes a conception of what geometry is. Consequently, mainstream theories are difficult to be applied for our laboratory, because of our conception of geometry (above) - in particular, of our blurring the distinction between Euclidean and non Euclidean geometry/ geometries.

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[13] A space so endowed is known as frame-space – and it does not require coordinates (Noll, 2004, p. 6, 7).

[14] In our understanding, Pinker asserts that « reference frames » are as instinctual (genetically embedded) as the language, for which he discussed his thesis in Pinker, 1994.

We wish to add that we could not find in Piaget, the Van Hiele's or in Tall's work or in other work, the recognition of the prominence of the "system building" stage (see paragraph 3.3) – which we would set at the top of the ladder - as it will emerge from the analysis of Euclid's work. Even more, since system building is peculiar of all rationalizations of empirical physical sciences – as discussed in the paragraph, Euclid's geometry as a system, this stage features not just geometry or mathematical physics but large parts of "hard" sciences and can also be regarded as part of the general scientific concept of "model" – something that percolates down to our students middle school tier in the simple form of mathematization.

The centrality of the sensorimotorial experience implies that the learning experience of our students is strongly dependent on the kinesthetic and spatial intelligence of the multiple intelligence models (Gardner) – and, consequently, is necessary to explicitly account for the class heterogeneity by the pedagogical differentiation strategies.

Yet, the logico-linguistic intelligence[15] has a critical role, since it enables the processing of kinesthetic and visual cognition by logical thinking (inductive in the discovery phase and hypothetical deductive in the development phase). Indeed, even if the abstract operations stage has just started (Piaget), plastic arts and cognition studies show that the student visual processing in pre-adolescence: " […] manifesta un'adesione maggiore alle cose e alle situazioni così come queste appaiono nella loro specificità. […] la graduale messa a punto di un pensiero analitico-deduttivo soppianta sempre più la caratteristica visione sincretico-proiettiva del bambino" (Bianchi, 2015, p.19)

We believe that it is the dynamics between these two intelligences that is more impacting the effectiveness of our non Euclidean situations. By dynamics we mean the way by which Dynamic-Systems Theory (Miller, 2011, p. 414) look at "things". To provide a concrete, though oversimplified, example, we consider the interplay of the two intelligences during the building of the concept of curvature – as defined in this work.

On a transparent plastic sphere we have traced in red an arch of a meridian. We have understood it to be a straight line, for example, because it would be so for creatures living on the surface and that are so small to perceive it as absolutely flat. We carefully observe it:

[15] Differently from Gardner, that speaks of logico-mathematical intelligence, we believe that in our case it is more appropriate to consider the combination of logic and linguistic competencies, as it will become apparent in the whole laboratory.

1. in response to our kinesthetical intelligence (KI), our logico-linguistic intelligence (LLI) ” tells us “that straight line is curving!”

2. KI responds by refining our feelings (we have to make it in words), “I do not feel that it is curving, I feel that it is the underlying plastic that is curved”

3. LLI responds by answering, “we have got to distinguish “two curvatures”: the curving of the line and the curved surface. The line inherits the curvature of the surface”

4. KI responds by reorganizing the perception and makes us feel more clearly “I feel that that line is really straight; there is some other actor: I feel the curvature of surface as separate”

5. LLI responds by reorganizing dialectically, “the straight line is straight. There is no inheritance. The curvature of space does not belong to the world in which the line lives. The line cannot inherit it. If the world were the space in which the surface is embedded, then the line would be only a piece of the surface and as such it would be curving. The line is also curving.

In our view, KI and LLI are both states of the system – but they are in retroaction loop (feedback system). System instability is associated to the conflictual states of KI and LLI. New stable configuration is reached by the mutual feedback. More importantly, as we will discuss: a weak LLI system prevents the KI system from reaching higher states.

The purpose of the example is to show that, in our type of cognition conflicts, it is by a tight interplay of the two intelligences that the synthesis is reached.

-> Therefore, the logico-linguistic intelligence, although not radically central as the kinesthetic, it is critical to process those cognition conflicts in which new images and models that contradict the preexisting ones have to be accommodated (Piaget) – a situation characteristic of non Euclidean aspects.

We note that we could have reexpressed all Piaget’s references in term of dynamic-systems “concepts”. We did not do it because we believe that the dynamic-systems concepts are not so widespread and would make the communication obscure.

Yet, we use these concepts everywhere: from the general relativity equation[16] and the discussion about how this chapter was built in feedback with the following, to all our (participating) observations of students. We may add, that an additional reason for the difficulties we have with the model of Tall, of the previous figure, is that its main structure is a "multi-tier static stack". In our opinion, our situations are better modelled in terms of "loops" between what in Tall's schema, is inside the tiers (further to have a radically different idea for the epistemology of geometry – as discussed).

Cognitive conflict, a key cause of system instability, is actually essential to mathematical learning. Indeed, as underlined in D'Amore (1999,) two fundamental aspects appear to characterize the process of mathematical learning: it proceeds by stages and by cognitive ruptures.

-> Therefore: "cognitive conflicts" are not to be avoided, but engineered in the laboratory – they are the "instability zone" by which the student ramps up in his learning process

Indeed, we have threaded logico-linguistic riddles to bring students to break down their existing images (we have called them, empathic quiz's, for they act on student's empathy capability).

On the other side, it is well known that the socio/ emotional/ affective aspects have to be an explicit subject of design, planning and monitoring – for they play a fundamental role in learning in general and in overcoming the cognition conflicts (Zan, 2007).

For the language in general, we assume the point of view of Vygotsky, developed by Vergnaud (1985): language's function is not limited to social communications, but extends to structure our thinking – which for Vygotsky, begins with the "private speech" and develops later in "inner speech" (Miller, 2002, p. 185-187 and p. 195).

In the line of thought of Vygotsky we think of the cognitive artifacts (Luria, 1976 and Normann, 1993) embedded in the social-learning model of Vygotsky – in particular, with the basic ideas of "Zone of Proximal Development" and "internalization". In our laboratory, kinesthetical cognitive processing takes a large part of the student's work and the semiotic process that enables the internalization is much less based on natural language than in traditional laboratories.

With respect to the relation between the approach of Piaget and of Vygotsky. We see them complementary, not in conflict, in agreement with Cole & Wertsch, (1996).

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[16] Our view is that of mathematical physics: dynamical systems were born when Newton and Leibniz wrote the first differential equations – the first mathematical models capable of capturing cause-effect relationships in space and time. Cause effect models are the kernel of mathematical physics.

Finally, we wish to note that in contrast with a conception of geometric cognition as cognition of physics of movements, the school traditional geometry privileges the study (much less the acting) of the static perspective. Kinematics and dynamics are left to physics[17]. This might be acceptable for Euclidean geometry, because the student has already developed by himself in daily life the "rest" of geometry - even before going to school[18]. Obviously, for non Euclidean geometry, it "would not work", because the student does not have an equivalent sensorimotorial experience and kinematics of non Euclidean phenomena.

It is our opinion that common strategies of proposing non Euclidean geometry at the level of static geometry, or as a formal system, or as experiments on an artifact not changing the overall perception of the world-space of the student, do not address the issue. As a result, for the student there will be "real straight lines" – the Euclidean, well rooted into the "whole geometry", and the "fake straight lines" – the non Euclidean, sitting as curiosity on top of his geometry models.

2.2 Synthesis

Effective learning of (non) Euclidean aspects requires working at the ground level of the student's geometric cognition – that for us is physics (of movements). With respect to our environmental conditions and to the authors and theories we have selected for discussion and our understanding of them – we believe that the essential guiding lines for our laboratory should be:

1. body centric – the student experience of geometry is centered on his acting by his body, through position and sense of direction (orienteering or spatial sense [19] – knowing where one is and how to get around), in contrast to traditional school geometry in which the student body is a non-participating actor.

2. sensorimotorial – the student geometric cognition builds on sensorimotorial perceptions of his body and of the observed objects, ordered in his subjective "space" and "time"

[17] As it was for in science before Einstein eventually managed to bring them into geometry of gravitation.

[18] As argued, also by Dewey and Montessori.

[19] In this work we do not generally distinguish between spatial sense, spatial thinking and orienteering.

3. kinematics and dynamics – cognition of space, forms and figures is kinematical and dynamical (related to experiencing "forces") - in contrast to traditional school geometry focus on static images and models

4. straight movement – it is built-in the primordial cognition of the student and is essentially kinematical and dynamical

These four requirements could be seen as "embodiment of a (mathematical) physics of (relativistic) movements"

5. geometry of his world (authenticity) – geometric cognition's subject is the student's physical world – in contrast to abstract mathematical geometry

6. synergy of kinesthetic and logico-linguistic intelligence – the second is essential to process concrete discoveries about the world and to reorganize the world perception and models

7. integration by restructuring – learning happens by integration; integration starts from recognizing the existing knowledge; discoveries impact on existing knowledge by assimilation or accommodations (cognition ruptures and conflicts)[20]

8. learning is "individual" – the large number of cognitive axes that characterize learning amplifies students' diversity and attention to differentiation

Finally, but first as pedagogical and educational concern,

- the entire work must provide the student with an experience that is meaningful with respect to his personal interests and of his group, with respect to the disciplinary objectives for his grade and with respect to the conception of mathematics and science and their role in the vision of society, culture, education and values that the school system aims to convey (see for example: Frapolli & Sbaragli, 2012).

[20] We found this formulation, of Piagettian inspiration, more suitable for our basic situation, than other more recent, like: Tall's based on his "set-before" and "met-before".

Chapter 3: A short and focused historico-epistemological analysis

This chapter is the result of a focused work: extracting epistemological threads and patterns from our understating of the basic major historical facts as reported in secondary literature (apart some exceptions). We propose many "conceptualizations" – that we will use as raw materials to assemble, in chapter 4, our epistemology to respond to the psycho-educational requirements. We endeavored to explain all main ideas without "formulas" – except one most notable exception in general relativity.

Gray (2008) provides, in our opinion, a very synthetic, but rich, historico-epistemological account of the interrelationships of the geometries that we discuss in the following,

3.1 Euclid's geometry five postulates

In this paragraph we report the original formulation[21] of Euclid of the V postulate with some of the surrounding foundational elements. Note that for Euclid a straight line is a segment or a ray or a segment indefinitely extended on both sides. The text of reference is Heath (1956), where we have:

"EUCLID's BOOK 1

Definitions

1. A point is that which has not part

2. A line is breadthless length

3. The extremities of a line are points

4. A straight line is a line which lies evenly with the points on itself

5. A surface is that which has length and breadth only

[definitions from 6 to 23 follow and are not reported here]

[21] As it has come down through history and in the translation of Sir T. Heath.

Postulates

Let the following be postulated:

1. to draw a straight line from any point to any point

2. to produce a finite straight line continuously in a straight line

3. to describe a circle with any center and distance

4. that all right angles are equal to one other

5. that, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles".

The first postulate means more than it could appear to a modern reader - understood in its context, it means that, ideally, straight lines between points do exist (in the sense of Aristotle) and that there is exactly one straight line joining two points (i.e., two straight lines joining two points "do not enclose a space").

The fifth postulate, in all generality, can be replaced by the following postulate of uniqueness – generally referred as Playfair's Axiom, but known already to Proclus: "given a straight line and a point out of it, there is no more than one straight line through the point and parallel to the line" (Heath, 1956, p. 220).

The existence is already a consequence, with no other assumptions, of the first axioms – and in fact it is stated by Euclid, in Heath, 1956, p.315:

"Proposition 31

Through a given point to draw a straight line parallel to a given straight line".

Yet, the most popular formulation of the V Postulate is

"given a straight line and a point out of it, there is exactly one straight line through the point and parallel to the line" (existence and uniqueness)

This formulation renders the system brittle, for when we negate this formulation of the V postulate, the other postulates are impacted and the system becomes inconsistent (see the paragraph, Euclid's geometry as a system).

The formulation may be used in discussion of non Euclidean geometry, but remaining aware that we cannot simply deny the existence part of it without changing the other postulates (as it will be shown in the paragraph, Non Euclidean geometries as per their origins).

3.2 Euclid's geometry deep foundations

Men of all times and cultures have developed a "know how" about what we today call "geometrical aspects" of land, heavens and the objects therein. In so doing, they have used or emphasized to different degrees their kinesthetic, visual, logico-linguistic and general abstraction faculties, as well as, in some cases, mystical or spiritual attitudes – as in the middle Orient and by the Pythagoreans. Based on contemporary history of Greek philosophy and civilization, for example, in Kraut, 1992, and McKeon, 1947 - and on the key insights of modern philosophy, for example, in Heidegger, 1927 and Naess, 1968, we would say that, sometimes between the 6th and the 5th century BCE, the Greeks developed what we would call here a "rationalistic attitude" towards some "questions": *what is "that" which is?* (metaphysics); *what is it -"is"?* (ontology) and *what is "knowledge"*? (gnoseology). Within that Greek world, that made of the art of argumentation a pillar of their culture, as exemplary displayed in Socrates' dialogues, Plato and Aristotle transformed that art into a science (epistemology) based on a systematic theory of knowledge (gnoseology) onto which mathematics and geometry were founded, but, of which, they were themselves a (psychological) root, in the sense that, the compelling character of geometrical "truths", already laid out by the theory of geometrical proof by Pythagoras, was inspirational to believing that a theory of knowledge would, indeed, be a possible and worth endeavor. We metaphorically look at the Euclid's Books as the product of that culture, the tip of that iceberg, or better, cathedral.

Before looking at the tip, we pierce into some details of the cathedral, because they allow us to understand "what was (the meaning of) Euclid's geometry" for the men of science up to XVIII century and why it is not surprising that nobody before Gauss could not think of considering Euclid's geometry without the V Postulate.

Mathematics for the Greeks was true knowledge, i.e., universal and necessary. Indeed, the deductive character of the mathematical proof, invented by the Greeks who attributed it to the Pythagoreans, was looked at as a compelling example that true knowledge can be attained. It provided a solution to the apparent chaotic flux featuring the physical world. Though some Greeks objected to this position, most notably the Sophists, who regarded knowledge as know how or as effective practice, the rest were in large agreement and differed only regarding the process by which knowledge is attained – with Plato and Aristotle representing the two major streams.

Plato attributed an “important degree of reality” to mathematical objects and their properties. In his philosophy, he consistently distinguished four different levels of reality and of knowledge of it (Kraut, 1992):

1. the two lower levels, in which there are physical objects like a baseball, are about the reality of the senses - a reality in flux, the understanding of which is particularized, contingent, temporary and probable, and not leading to true knowledge

2. the two upper levels, in which there are mathematical objects like a sphere, are about the reality of thoughts - an immutable reality the understanding of which is universal, necessary and certain, and leading to true knowledge.

Plato agrees that mathematical objects do exist as products of our mental modelling of the physical world, but, adds, that they also exist independently and before our modelling. We apprehend them by hypothetical-deductive reasoning, in which, for example, the triangle angles property (the sum) is true to the degree to which the postulates and common notions of geometry are true. Yet, notice, that mathematical objects are not at the top level – where there are forms and “ideas”(e.g., “the Justice”, “the Good”) that are apprehended by dialectic. Working on mathematics is a way to get trained in dialects.

Aristotle’s view is different. Knowledge of mathematical objects is still true knowledge, but the objects of mathematics are (just) abstractions from physical objects. We have an “intuition faculty” helping us to get “right” the fundamental objects and facts of geometry (those to be construed into postulates, common notions and definitions) – whereas, for sciences we induce them from experience and bridge them with intuition. The forms for Aristotle are forms of matter, not the Forms of Plato, and triangles and circumferences are abstractions from empirical observations, not “things” more real than the corresponding physical objects (McKeon, 1947). Aristotle does not follow his master Plato scale of degrees of reality. Aristotle studied proof based systems of geometry as the one of Hippocrates of Chios – the only one of which we have notice predating Euclid’s. Aristotle looked at the postulates at the basis of the hypothetical-deductive reasoning and thought they should be self-evident truths. How do we get these premises right – self-evident? Aristotle says, by inductive reasoning and “intuition”: inductive reasoning (from particular to general)[22] is applied to the physical world and empirical facts to produce the “raw materials” to

[22] Inductive reasoning, from Aristotle to Goodman, through Bacon, Hume, Comte, Mill, Reichenbach and Popper - to name a few, it has been a central topic of discussion for science and gnoseology.

which our intuition faculty ("intuitive knowledge of universals") is applied to get to true premises - at which point, hypothetical deductive reasoning is plugged in to arrive to complex and true knowledge. An example that we think may help is: at all times and in all places where there were taken measures of the length of the circumference and of its diameter their ratio appeared to be approximatively the same independently from the diameter (empirical data); we induce that at any past and future time and at any place a circumference of any diameter has its length and the diameter in the same approximate ratio (induction); we mathematize this induction by the intuition that the ratio must be one and one only number and we call it "π" (at which point we have an intuition of a "universal") – and we select as best approximation of it the value 3,14; we prove that if the equator of a sphere is 40'000 km, the length of a ring tracked by an apple orbiting around the equator at an height of 1 m is 2π m, that is approximatively 6,28 m, longer than the equator (deduction).

Historians of science agree that the science of Aristotle, that dominated up to the XVIII century, looked at geometry as true knowledge (in contemporary language we might call it absolute), attained through inductions from empirical facts fuelling the intuition of the "universals" and followed by deductive inferences.

A note

There are some scholars, for example Migliorato, that argue that the Euclid's geometry as a "completed work" ("come si presenta l'opera finita [di Euclide]", Migliorato, 2005, p. 18) appears to be more a work of pragmatist mathematical modelling than a part of a theory of knowledge ("[…] la formalizzazione idealizzata (modello astratto) di "un fare" (come pratica di risoluzione di problemi) piuttosto che di "un essere" (altro che Euclide platonico!)", Migliorato, 2005, p. 22). Although we recognize the value of the punctual and analytical arguments brought to sustain this thesis in Migliorato (2005) – we do not believe them to be in line with an hermeneutics netted from our contemporary scientific mindset. Indeed, contrary to what Migliorato (2005) assumes, we believe that the paradigms of "problem solving", "mathematics as pragmatic model based on its success in making sense of empirical results", are paradigms of our modern Weltanschauung (world view): it is because we have a certain modern scientific Weltanschauung that we can conceive these paradigms, that, in our opinion, are alien to the Weltanschauung of ancient Greek civilization. That said, Euclid as an individual might have been an "outsider". Yet, what the work of Euclid was for all other men is what was compatible with the Weltanschauung of the time.

Based on our overall discussion, and contrary to some widespread opinions, our point of view is that:

1. Euclid's geometry is not a collection of long known empirical geometrical facts discovered by previous cultures or Greeks: it is the tip of a cathedral of knowledge

2. Euclid's geometry was not a formal mathematical theory decoupled from physical world or knowledge about it

3.3 Euclid's geometry as a system

In the previous two paragraphs we have looked at Euclid's geometry as "geometry statements or contents" and as "knowledge about the world". Now we look at it for its "system structure" – for it is a complex knowledge management system.

This system perspective will provide us with a different point of view to assess to which extent and in what sense non Euclidean geometries and Riemann's geometry deviate from Euclid's.

This perspective will also allow us to highlight some value that Euclid's geometry has had for men of all times and sciences, gain an insight into the core process of mathematical physics building, underline some implications for development of students mathematical and scientific skills and, finally, contextualize Euclid's system in modern axiomatics.

The schema of the next page illustrates our deconstruction of Euclid's work[23]:

[23] "Euclid" stands for a virtual actor of the process, not for the historical character

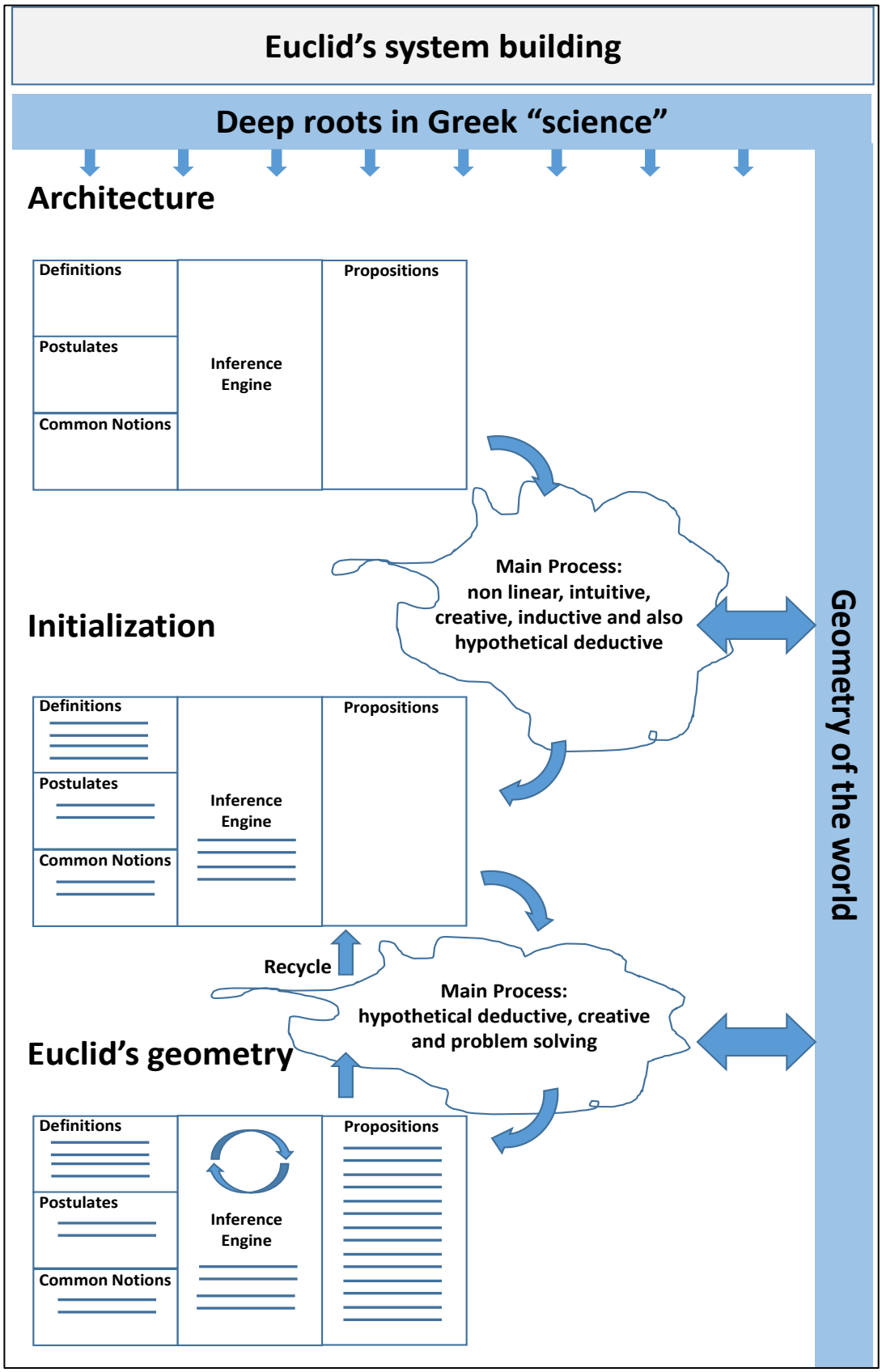

Euclid's system building
Deep roots in Greek "science"
Architecture
Definitions
Postulates
Common Notions
Inference Engine
Propositions
Main Process: non linear, intuitive, creative, inductive and also hypothetical deductive
Initialization
Definitions
Postulates
Common Notions
Inference Engine
Propositions
Recycle
Main Process: hypothetical deductive, creative and problem solving
Euclid's geometry
Definitions
Postulates
Common Notions
Inference Engine
Propositions
Geometry of the world

Figure 3. 3

A process in three phases

Phase I. Euclid takes the motherboard of the science of reasoning Aristotle – induction and deduction plus the intuition faculty[24]

Phase II. Euclid, by mastering the art of system design and considering all empirical known geometry, through a nonlinear and complex inductive and intuitive process comes to a consistent set of definitions, postulates and common notions from which he goes about building by deduction all key results (the propositions or theorems) which all together are a faithful image of the world as we perceive and understand it

Phase III. Euclid polishes the hypothetical deductive edifice, takes off all the rest and consigns it for posterity

An output structured in four "blocks"

A. architecture (of the system)

1. Three sets: definitions, postulates and common notions
2. Inference rules: appropriate deductive rules from Aristotle's set
3. Propositions (Theorems): to be generated by applying appropriate sequences of inference rules to the definitions, postulates and common notions

B. primitive contents: what we put in (initialization):

1. the actual choices he made for definitions, postulates and common terms
2. the inference rules he accepted (for example: he accepted proof by contradiction, which is not always the case for some modern mathematics)

C. derived contents: the generated results (producing and proving theorems)

[24] We propose that one "success critical factor" specific to Aristotelian science of reasoning or logic-system, is that it lends itself well to deal with our daily physical experience of the world – contrary to other "logic systems", for example, the paradoxical logic that we find in Heraclitus, Lao-Tsu or some Indian logic, that have a different aim: let us "understand" (what the science would call) the mystical or spiritual world. Statements like: "That which is one is one. That which is not-one, is also one" (Fromm et Al., 1960, p.101 and 102), though enlightening for a mystical knowledge, cannot be used to build an "inference system" for an hypothetical-deductive theory of geometry that works in our daily experience of the physical world. The Euclid's system, and its inference engine, did not presuppose just a science of logic, but a science of logic effective with the physical world.

1. the propositions (theorems) iteratively generated by applying the inference rules to postulates - and using definitions, common notions

D. criteria for truth and optimality

1. Euclid system is about all geometry that is true in the world
2. from the way the propositions appeared to be engineered one in relation to the other, we may assume that Euclid applied all Greek science for "good argumentation" to design an optimal system

The process and the schema let us appreciate some commonly overlooked value of Euclid's achievement. When we consider how carefully Euclid's has managed to decouple the V Postulate from any other element, we may appreciated how complex is to accomplish a B1 consistent with C1 and D2, while at the same time, accounting for all (empirically) known results at the same time, D1.

Let use the system perspective to anticipate to which degree non Euclidean geometries and Riemannian manifolds geometry differ from Euclid's:

1. What is hyperbolic geometry about? Changing one "lever", one initial setting of the machine – the V Postulate, and apply deductive reasoning to regenerate consistently all results. In the historical context, an act of "intellectual boldness", and also a large deductive work where intuition could not help (there was no world outside to look at for inspiration!) – but, little of the work done by Euclid in phase II above and no architecture reworking
2. By doing so, the traditional non Euclidean geometries cut the system out of its "sense" foundations: the world and the theory of knowledge of the world or any pragmatic (if we adopt the view that Euclid's geometry is "just" a mathematical physics model of the world) – yet, they strictly move inside the "system" of Euclid
3. Riemann, instead, put aside Euclid's system, but not the theory of proof and logic (inference engine), and builds his own system, analytical and differential[25].

This view of the Euclid's geometry, in our opinion, clearly explains the reasons of the educational value and success of its "Books"– for man of all times, cultures, classes, professions and sciences. Indeed, Euclid's system building was done for geometry, but nothing forbids it from being done for

25 Differential does not imply analytical: there are synthetic differential systems like Synthetic Differential Geometry, Koch, 2009 – see also in Attachment 4, the Challenges section.

other fields. There are countless examples in all fields. Leaving aside all mathematics that have been structured axiomatically, we may recall the "Ethics" of Spinoza (Spinoza, 2001), in which an entire Ethics is derived from a few assumptions; the "Declaration of Independence" of the United States, in which some initial truths are postulated and from them the declaration is derived; the "Mathematical Principles of Natural Philosophy" of Newton (Newton, 1999), beginning with definitions and axioms or laws of motion and going about drawing implications for three books.

In management, economic theories and natural evolution theories, for example – it is common to establish some basic principles, state some constraints or conditions, and then going about deriving implications by using an Aristotelian logic. In mathematical physics, starting from analytical mechanics and rational mechanics, to rational thermodynamics, fields' theory, and of course relativity theory - to different degrees of formalization and with different lexicons, we find basic postulates and definitions, inference rules and derived results. But what is more important in our opinion, is that the two main processes of our figure are at the basis of the modern construction (systematization) of all physical mathematics.

We think that this perspective take mathematical education attention to the importance of the "system building" component of mathematical (actually scientific) learning. As we will discuss at the end of the paragraph this aspect is missing in the Van Hiele's. Indeed, in this work, as it will be discussed, the epistemology of geometry, in particular, non Euclidean, is shifted from the dominating purely mathematical view, characteristic of the traditional non Euclidean geometries, to a (mathematical) physical epistemology of movements (in a curved space where the general relativity equivalence principle holds) – for it is that latter, we propose, that allows us to overcome traditional opposition of Euclidean and non Euclidean geometries, not in abstract sense (as in the common interpretation of Riemannian manifolds), but in the world of the student (see also the more general view expressed by Arnold, in paragraph 3.8).

We briefly come back to the system of Euclid, to discuss the criticism that was made to its "soundness". Indeed, its mathematical and logical integrity is far from meeting modern standards – as they result from a process started at the end of XIX century. The revision of the foundations of mathematics and logic, in particular the work of Russell, Hilbert and Gödel, showed not only the weaknesses of the Euclid's specific system choices, for which several new axiomatic systems were proposed (see for the Hilbert's example, Arzarello et al., 2012) - but the existence of "a priori limitations affecting any axiomatic/ deductive system of mathematics". Yet, in our conception of mathematics and of its learning, socio-constructivist, we do not regard these formal limitations as impacting on the value of the system as reference model for constructing human knowledge. On the

contrary, we look at them, accordingly to Rorty (1979), as therapeutic, for they dispel illusory pretensions to absolute truth and completeness and let us focus on the success that the Euclidean geometry, but also many other parts of Physics, had, and continue to having, in allowing us control on the world of our experience.

Based on the above, we have found that common discussion about Euclid's work, miss one or several of the following points:

1. Euclid's geometry is not just a collection of long known empirical geometrical facts discovered by previous cultures or Greeks: it is a complex knowledge management system
2. The system by which geometry is organized does not consist only of interdependent propositions: there is an inference engine built on the Greek theory of logic and knowledge
3. The craft of building the system presupposes much more than hypothetical-deductive reasoning, for it includes: mastery in observing and analyzing the world, inductive reasoning and optimal system design; only the combined use of all these processes makes it possible to build science as a system (system building)
4. It is unlikely that by an epistemology of pure mathematics that is divorced from its foundations or relationships with physics, the students will develop into "prototypes of scientists"
5. The system is (was) not a formal system: it does (did) have "meaning"
6. The traditional non Euclidean geometries, as we will argue furthermore, are not a cut with Euclid's system: they only cut the system roots with the "world" and "switch a lever"
7. Riemannian manifolds, as we will argue furthermore, are a cut with the Euclid's system – but of a very different type

A final educational observation

With regard to the third item, it is significant, in our opinion, that the van Hiele's model ignores the prominence of the system building – it is not even a stage of their scale. We have found no mainstream mathematical educational work on geometric cognition in which the previous "system building" activity (item 3), is explicitly recognized as a principal development stage – which we would put above the universally recognized hypothetical-deductive and rigorous formal stages (stage 4 and 5 of the Van Hiele's and top level of Tall's multitier model presented in chapter 2). In our opinion, this might be an effect of two factors:

a.it seems to us that, in the mathematical education sector, the more pervasive "conception of the body of knowledge of mathematics" is that of "pure mathematics" instead of "mathematical

physics" - a conception which we have hinted to as underlying also Piaget's work in chapter 2. This conception makes difficult to identify or accept as central to mathematics the system building process that we have proposed central to the conception of "mathematical physics" and to the work of Euclid.

b. in our opinion, there is a lingering (fossil) effect of a traditional school in which geometry, as many other subjects, was supposed to be learned by "being transmitted" - not as we would like, by "being constructed", which leads to the climax of system building.

3.4 Non Euclidean geometries as per their origins

We review very shortly the origin of the non Euclidean geometries - underlining their act of birth as a declaration of independence of geometry from world's geometry, their consequent formal character, that "paradigm of Many Geometries" that they originated and how Riemannian manifolds are commonly "assimilated" within that paradigm as more general non Euclidean geometries.

Our review is a fast lane to motivate our epistemology based on Riemannian manifolds and general relativity. For more informative accounts of non Euclidean geometry, see: Greenberg, 1994; Bonola, 1906, and Agazzi & Palladino, 1976.

Everything starts from the V Postulate of Euclid, and its equivalent formulation:

"given a straight line and a point out of it, there is no more than one straight line through the point and parallel to the line" (uniqueness)

The postulate concerns the uniqueness of the parallel straight line - its existence is already implied by other postulates (see previous paragraph, Euclid's geometry five postulates – the comment on the V Postulate).

Since the times of Euclid's himself, the V Postulate was found unsatisfactory for different reasons, including (Heath, 1956): it did not appear as self-evident as the other postulates; it was not sound for it made assumptions on something that one could not go and check even for one (!) case (for how many couple of parallels has anyone ever checked that they do not meet if indefinitely extended?); it seemed something that could be derived from the other postulates and theorems (Euclid built the system in a way that the first 28 theorems do not need the V Postulate – another evidence of his or his team system design mastery).

What was not questioned, instead, was its necessity: it was only its formulation that was perceived as not optimal.

After the fall of the Roman Empire and the rediscovery of the Greeks by the Islamic world - in particular, the "Philosopher" (Aristotle) and Euclid, the quest and work on the V postulate were revived and spread back to Europe, and in the XVIII century, with Saccheri, to a different approach for proving the V Postulate (1733). Saccheri aim still was to prove the V postulate, but he meant doing it by "a proof by contradiction" – the logic of which had already been systematized by Aristotle in his Logic (reductio ad absurdum), but used by the Greeks since before Socrates – who would typically use it in his Dialogues to bring his "opponent" to fallacies deriving from his own assumptions and, thus, making him abandon his convictions or at least doubt of their validity. The proof by contradiction had already been used in the work around the V Postulate, but never as directly as by Saccheri to prove the V Postulate.

Saccheri assumes that the V Postulate be false (in its original formulation) and infers theorems based on this assumption and the other postulates. He correctly infers that the sum of the internal angles of a triangle is less than a flat angle, that parallel straight lines are not everywhere at the same distance, and other similarly striking theorems. Saccheri regards these results as "repugnant". This should not be surprising: Saccheri is reasoning in terms of "meaning" - as all great men of science before him had done, and as our students do. Geometry was not a formal system, it was the tip of a cathedral of knowledge about the world (see paragraph, Euclid's geometry deep foundations). If Saccheri had had a conception of geometry as model of the world, he would not have found the implications repugnant "enough to build whatever possible means to refuse them"[26] – he would have found them not corresponding to that part of the physical experience he had of the world.

The shift of paradigm that Saccheri fails to make is done by Gauss nearly one hundred years later (around 1813[27]): for him, it is not a question of making sense, i.e., it is not a question of fitting geometry in our worldview, but it is a question of being or not logically valid (Greenberg, 1994 or Agazzi & Palladino, 1976). In our metaphor, Gauss does not look at geometry vertically (as a tip of the cathedral). He looks at it "horizontally" – as a system. In this mindset, the Saccheri theorems were indeed logically valid consequences of the other postulates. But Gauss keeps it to himself. In

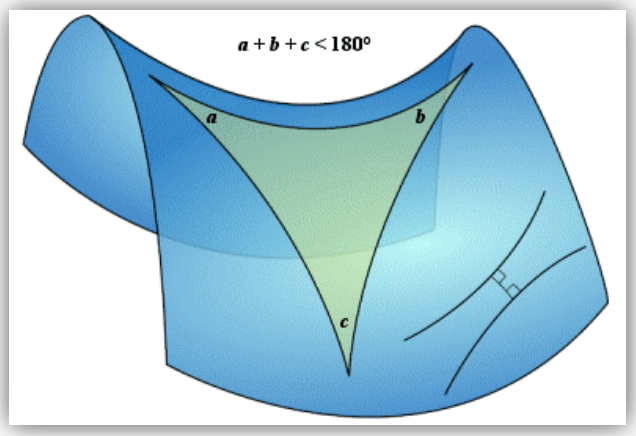
[26] Indeed, he even ventures in the emerging infinitesimal analysis, making mistakes, in refusing the non Euclidean implications by (Agazzi & Palladino, 1976, p.76)

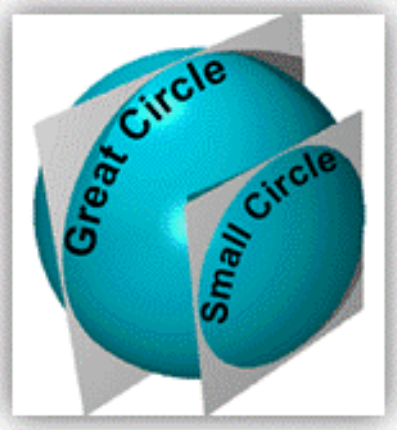
[27] See Burris, 2003

the German scientific community dominated by the philosophy of Kant, who had stated the Euclidean geometry as unquestionable, it is understandable that Gauss, when later asked about why he had not published his discovery and left Bolyai (and Lobachevsky) to be first to "publish" their discovery (around 1830), Gauss answered that he did not do it, because he was fearing the "yelling of the Boeotians".

Bolyai and Lobachevsky independently from Gauss, and only a few years later, one in Hungary and the other in Russia, explored the geometry obtained by denying the V Postulate and wrote their accounts of the results so obtained. Lobachevsky even built an entire system out of it. Bolyai called its "the absolute science of space". And Lobachevsky, "Imaginary geometry". It was Gauss who called it, non Euclidean[28].

The geometry stemming from these works is called "hyperbolical". It is a 3D abstract geometry. Yet, its 2D version can be "modelled" by 2D surfaces[29]. Modelled means that, if we impose the name of "straight lines" to some specific curving lines of these surfaces (to simplify, minimal distance paths), then these "straight lines" behave not as Euclid's straight lines, but accordingly to the axiomatic of the straight lines of abstract hyperbolic geometry.





Figures 3. 4

[28] Yet, from our system point of view, we can see as much of Euclidean as of non Euclidean in it – in contrast to Riemannian manifolds.

[29] Beltrami was the first to propose a visualization (local, the tractrix). Poincaré provided a later one (global, hyperbolic disk).

The spherical[30] geometry corresponds to having no parallels. This is obviously not a question of the V postulate – that concerns uniqueness. It is necessary to drop the first and third postulate (see paragraph, Euclid's geometry five postulates) – so that it cannot be proved that there is at least one parallel. This geometry is as abstract as the hyperbolic, but likewise, its 2D version admits a 2D model– if we attach the name of straight lines to the great circles of a sphere.

At this point the key shift has begun: the geometry that Euclid has left in the Books is regarded as a system based on logic, the validity of which can be conceived in formal terms with no need for backing it up by the physical world (no need for meaning, only for correctness). The idea of a mathematical geometry divorced from the world's geometry is born, and, as a side effect, the problem of "which one is the geometry of the world?".

Yet, on the mathematical side, things are all, but on safe grounds: "How to be sure that a geometry obtained by denying the V postulate is logically consistent?" - is the first problem to be faced. A problem of which, it was then realized, was suffering the Euclidean geometry itself - for no one had ever taken the burden to check it out, may be because that had been backed up by two millennia of use and by the world-geometry, which had been taken to be "one-to-one with the geometry". Yet, in the new situation, this was no longer sufficient. It took the work of Beltrami, Klein and Hilbert (see, Greenberg, 1994), to provide arguments for the internal logical consistency of the geometries. In so doing, as we have already mentioned in the previous paragraph, Hilbert re-founded Euclid's geometry on a more logically sound system, in which any reference to intuitive ideas is casted out of the edifice and "things" are defined "by the relationships into which they participate" (in contrast to Euclid's geometry that had never been intended to be a self-sustaining system with no reference to the world – it rested on postulates not axioms[31]).

This formalistic direction of development would have eventually led to the work of Gödel and to the formulation of the essential limitations of the logical systems and, to a formalism that was later considered excessive and criticized by many scientist (see paragraph, V.I. Arnold and R. Thom on formalist mathematics).

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[30] We avoid more precise definitions of elliptical or hemispherical or Riemann's (which require working in the Hilbert's axiomatization of Euclid's geometry), because not worth to our objectives.

[31] As a rule, we have used the term postulate when the system is built on "meaning" (points to a world) and "axiom" when the system is formal. Consequently we never call Euclid's postulates axioms.

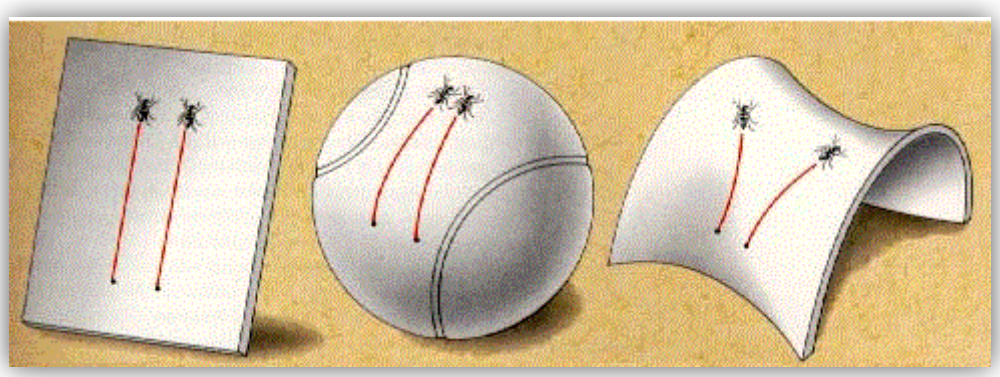
We note that both spherical and hyperbolical geometries are commonly said to derive from the denial of the V postulate. As explained, spherical/ elliptic geometry is not obtainable by denying the V postulate – but by denying the first and the third. The misunderstanding stems from considering the following popular (but nonequivalent) formulation of the V Postulate – that, as remarked in the paragraph, Euclid's geometry five postulates, in the comment on the V Postulate, makes the system inconsistent (brittle).:

"given a straight line and a point out of it, there is exactly one straight line through the point and parallel to the line" (existence and uniqueness)

From this statement, spherical and hyperbolic geometry are derivable by denying, respectively, existence and uniqueness.

With the work of Riemann in geometry, non Euclidean geometries were reframed as special cases of Riemannian manifolds (see Greenberg, 1994) as follows:

1. non Euclidean spherical geometry: a Riemannian manifold with constant and positive curvature
2. non Euclidean hyperbolic geometry: a Riemannian manifold with constant and negative curvature
3. the Euclidean geometry: a Riemannian manifolds with no curvature
4. Riemannian manifolds are seen as additional geometries
5. Riemannian manifolds are seen as a theory of all geometries (instances)



Figures 3. 5

This is the way by which common non Euclidean geometry regards Riemannian manifolds. In our epistemology we simply "turn it around" – starting in the paragraph, Riemannian manifolds.

We mention that non Euclidean geometries may also be discussed in projective geometry, where the Euclidean case is referred, generally, as parabolic. We do not discuss this point of view, extensively studied by Klein, because it is not functional to our objectives. For similar reasons we do not

discuss non Euclidean geometry through complex analysis (see for example, Needham, 1997, for an approach highly visual).

As a final point we discuss if it is true that non Euclidean geometries' systems our less optimal with respect to their ability to represent the geometry of the experience we have of the world. At first sight, the non Euclidean changes made to Euclid's postulates appear unnecessary and costly – forcing us to add ad hoc hypothesis to explain why we do not observe non Euclidean behaviors in our world. In other terms, they are contrary to Mach's principle of economy[32]: scientists must use the simplest means of arriving at their results and exclude everything not perceived by the senses. Yet, although non Euclidean geometries require assuming that on large scale things go differently from how they go on our scale of experience, Euclid is exposed to the same criticism: he assumes that on large scale things go as they go on small scale. To make an example, the former assume that parallels will converge, the latter that they will keep running in parallel. Yet, in our opinion, we would call Euclid's choice more optimal, because: Euclid uses a "spatial scale induction", while the non Euclidean geometries do not refuse the spatial induction (that case would lead to the point of view of neutral or absolute geometry), but use a "spatial scale anti-induction": they assume that "on large scale things will go the opposite of how they go on small scale".

This attitude cannot find any justification if geometry of the world is conceived as static geometry – a static mathematical model. It is only when the conception of the geometry of the world is revolutionized by including kinematics and dynamics (force fields) and textured in space and time that that assumption will, surprisingly, acquire sense (in general relativity).

As conclusion, the shift from one and true geometry to many mathematical geometries (and one unknown world geometry), was made possible by separating geometry as a mathematical system from geometry as a feature of our (geometrical) experience of the world – the latter always assumed to be corresponding to Euclid's. This is the key issue for our students – for whom, meaning is attached to their (geometrical) experience of the world[33]: for them, the non Euclidean geometries, by definition, do not make sense.

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[32] Or any loose formulation of Ockham's razor principle.

[33] There are some subtle wordings we have made here: by saying "geometrical experience of the world", geometry (for our student) becomes subjective and local which, in turn, makes impossible to assert that the Euclid's geometry is true and, eventually, opens up to Riemannian geometry. All these aspects will become clear throughout the work.

3.5 Riemannian manifolds

The contents of this paragraph are further developed and structured into our epistemological model in chapter 4. Our references for modern general analysis of Riemannian manifolds concepts are: Berger, 2003 and Misner et al., 1973.

The "Habilitation"

When Euclidean geometry was empowered by Descartes and Fermat, who algebrized its implicit Euclidean space and integrated in it the Euclidean metric, nothing changed to its space. Euclidean geometry was even more empowered by calculus, and made into differential geometry by Gauss – taking the study of curved figures (surfaces) to a level impossible before, and, yet, nothing changed to its space.

As we argue, Riemann revolution of Euclid's geometry starts from recognizing that figures belong to an environment space[34], that their constructions and properties depend on the features of that space, that the relationships between space and its figures had never been clarified and studied by geometry. Indeed, he stated in 1854, at Göttingen, during his "Habilitation" lecture, "Über die Hypothesen welche der Geometrie zu Grunde lieg":

"It is known that geometry <u>assumes</u>, as things given, both the notion of <u>space</u> and the first principles of <u>constructions</u> in space. She gives definitions of them which are merely nominal, while the true determinations appear in the form of axioms. The <u>relation</u> of these <u>assumptions</u> remains consequently in darkness; we neither perceive whether and how far their connection is necessary, nor a priori, whether <u>it is possible</u>" (Riemann, 1854, p.1 – underlines are ours).

In our view, from Euclid, through Descartes, to Gauss geometry had been "geometry of figures". With Riemann, geometry becomes "geometry of the environment in which the figures live[35]".

To implement his project Riemann had to answer a key question: what is the character (or attribute) of space from which the entire geometry of its figures can be derived?

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[34] Space has been a subject of reflections since ancient times. Piaget himself worked on the formation of the concepts of space, time and speed on a suggestion from Einstein. We take an objective driven and simplified approach and endeavor to stay clear from the innumerable philosophical detours that should or could be taken.

[35] We have started with environment to make clear that it may not be the ordinary physical space. Indeed, in relativity is space time and in mathematics can be any abstract space. We will generally drop environment for the commonly used "space".

To answer, Riemann could start from the differential geometry of his master, Gauss – as he mentions in his Habilitation.

Gauss had studied figures of ordinary three-dimensional space using infinitesimal calculus on surfaces (and volumes) to mathematize the geometry of the figures on the surfaces (and volumes). The key point was how to define distances "within" the surfaces. Gauss idea is using differentials (the Leibniz's infinitesimal change) to add them up in infinitesimal straight lines on which to calculate the distance by calculus.

In his Habilitation, it is in this idea of distance function (metric) that Riemann identifies the "attribute" of space from which all the rest is derivable. By assigning in every point of his space the infinitesimal distances to every other infinitesimally nearby point, one would have implicitly given the entire geometry of the space. Mathematically, the infinitesimal linear element ds (Riemann, 1854, p. 5):

[…] is expressed by rectilinear co-ordinates, $ds = \sqrt{(dx)^2}$

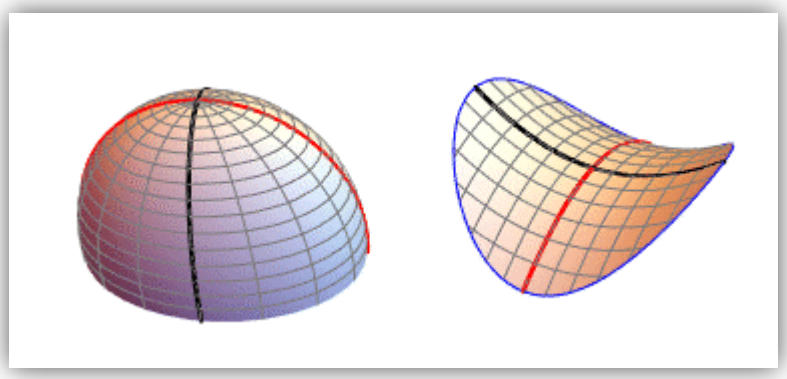
That is in contemporary notation, and for a three-dimensional space: $ds=\sqrt{(dx_1^2 + dx_2^2 + dx_3^2)}$: on infinitesimal scale, distances are given by the Pythagorean theorem – once the appropriate system of coordinates has been set.

Riemann generalized this metric to any number of dimensions. He noted that if the formula is true not only on infinitesimal scale, but on a scale however large, the resulting space is the ordinary three-dimensional space implicit in Euclid's geometry. We see that "Euclidean" has come to qualify a space – and the consequential geometry.

The spaces that Riemann is considering are on infinitesimal scale Euclidean spaces, but on larger scale they may be "as curved as one wishes"- as we will discuss now.

Gauss had studied the curvature of surfaces of ordinary Euclidean space, generalizing the common notion of curvature of a circumference as the reciprocal of the radius by finding a way to express the curvature of a surface that would not depend on how the surface was "bent" (embedded) in in the ordinary space. An A4 paper sheet may appear flat or curved depending on whether it is bent (for example, like a cylinder) or not (for example, lying on a table). On the contrary, there is no way to get rid of the curvature of a hemispherical shell by bending. Gauss managed to mathematize this second type of curvature (intrinsic). Gauss' curvature is +1 for a sphere, 0 for a flat surface and is

negative for a saddle shaped surface – which in a section curves “upward” and in another section curves “downward”.



Figures 3. 6

Riemann generalizes to n-dimensions Gauss mathematical expressions of curvature and their dependency from the metric. He uses what would later be formalized as a tensor and systematized into the “Calcolo Differenziale Assoluto” of Ricci-Curbastro and his disciple Levi-Civita – which Einstein used to build his Relativity Theory. The key fact is that the curvature is determined by the metric by some differential equations.

As it would later be shown by Weyl, metric is not essential to the concept of straight lines and curved spaces. Indeed, Weyl developed the affine differential geometry that was then further developed by Cartan who, building on the work of Grassman, led to modern differential forms and exterior calculus.

Manifolds

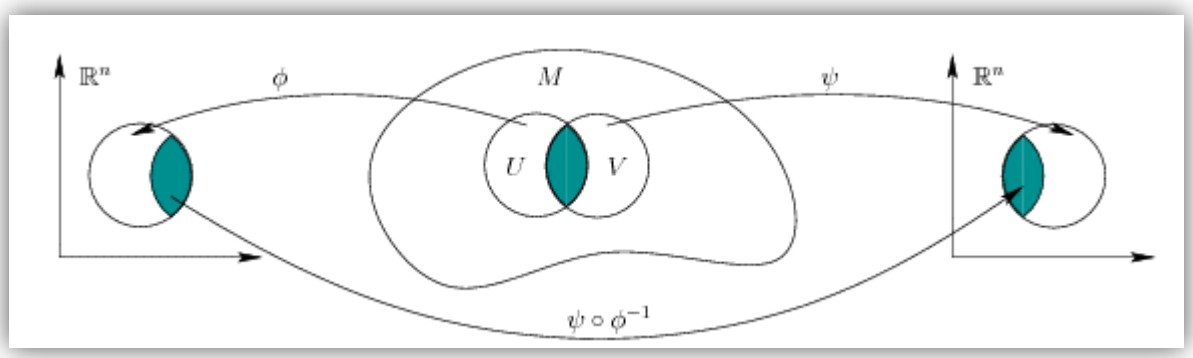
Riemann called his spaces, manifolds (Mannigfaltigkeit). Therefore: a Riemannian manifold is any multidimensional curved space that on a “small scale” is flat and with Pythagorean metric[36]. Flat and Pythagorean is commonly shortened in Euclidean: Euclidean = Flat & Pythagorean Metric[37].

Models of two-dimensional manifolds are curved surfaces of ordinary Euclidean space - like a horse saddle or a balloon. An image of a manifold, adequate for our purposes, is that of a smoothly

[36] There are flat spaces that are non Euclidean. It is indeed the case, and the most significant is the space time of special relativity which has a non Pythagorean metric: ds2 = cdt2 – dx2 – dy2 – dz2 (Lorentzian).

[37] Yet, some care is necessary - since Pythagorean is often replaced with Euclidean.

curved 2D surface, the geometry of which is lived from within, and that can be exactly mapped by its inhabitants on 2D infinitesimal flat maps.



Figures 3. 7

The term "manifold"

We had found the term "manifolds" confusing, because the word Mannigfaltigkeit and manifold point to the idea of "many" and let one wonder why to call "multi-whatever" something the most important feature of which is that of being curved. We have found that Kant (in Kant, 1928) before Riemann, discussed of multi-dimensional worlds and had called them Mannigfaltigkeit worlds. We think that, if his idea had become widely debated, then manifold would have simply come to mean multi-dimensional, flat, space – being that the only type of space known to Kant. Instead, Mannigfaltigkeit became popular only with Riemann's work – in which was used not for a flat multi-dimensional space, but for a curved multidimensional space. Thus manifold has come to mean multi-dimensional and curved.

Curved surfaces and traditional non Euclidean geometries

We believe a point is epistemologically central: a curved 2D surface is a manifold, if it is itself the space within which we do geometry, without leaving it. That means that it must be: "our environment space" – from which we cannot exit, and in which "Euclid's geometry does not work" (see previous comment about flat & Pythagorean metric).

Frequent statements are made that the study of curved surfaces of our 3D world would require traditional non Euclidean geometries or intrinsic treatment. In our opinion, this is not the case, for

we believe that we understand (cognition) curved surfaces from outside (the Euclidean environment space) and that they have always existed and have always been studied extrinsically (Gauss differential geometry builds some intrinsic elements from an extrinsic view). In addition, commonly known non Euclidean geometries are hyperbolic and spherical and do not apply to the study of other types of curved surfaces. In our view, if any alternative to Euclid's geometry has to be considered for a general study of curved surfaces, it is Riemannian manifolds (used in Euclidean embedding) – as it has already been the case, for example, in continuum mechanics.

The great shift

Riemann shift is: from "Geometry of Figures" to "Geometry of Space".

We have shown that Riemann puts in the center of geometry "its space" and defines its features by the "metric", which determines the curvature and the entire geometry of figures. There are as many geometries as metrics, including: Euclid's, the hyperbolic and the spherical non Euclidean geometries – further to many new non Euclidean geometries.

The great shift and curvature

We underline that the shift implies that, epistemologically, there is the "curvature of the space" and the "curvature of the figures of the space".

There is no "act of curving" for the former. Its "role" is bending the geometry of figures.

This epistemological and semantic point is a key psycho-didactical point for our laboratory – in which, from the start we have split the two curvatures for our students: the curvature of the space (for them a surface), as something "they feel the effects of" (as figures do); the curvature of the movements and lines, a "controllable" deviations from straight line. These curvatures do not sum up, for there is no outer or super space from which to look at their sum.

Comparison with Euclid's

The "shift" highlights how Riemann's geometry system is distant from Euclid's: Riemann is not concerned with postulates about the figures or their parts. He is concerned with postulates about space and he does it in line with "Zeitgeist" – the culture of his time: he makes assumptions about infinitesimals, metric, continuity, topology, etc. with knowledge of the "world" and of mathematics: inductively, intuitively and creatively. When, in his Habilitation lecture, speaks of general multidimensional spaces, it is never in speculative terms – he always refers to "magnitudes". It

appears to us, very similar to what we do with mathematics today, when we use it for financial markets or computing or any other application field: we have n-dimensional spaces representing the "variables" of the problem – they are "world" problem, though not "physical world's". Indeed, Riemann did apply his manifolds to world's geometry - as discussed more below.

For our purposes and in our view the key aspects of the Riemannian manifold concept are:

1. The environment space of geometry becomes an explicit object of study, the study is necessarily "from within" [38], and the space is called manifold – being generally curved and multidimensional
2. For Riemann[39], it is the metric that makes it possible the study "from within"; it depends on the place, it gives the rules to calculate distances in an infinitesimal neighborhood of any point, it may be considered mathematically as a tensor; pictorially it could be imagined as the fabric of the space on which distances are encoded at infinitesimal level
3. For Riemann[40], the curvature is derived from the metric, it depends on the place, it may be considered mathematically as a tensor; pictorially could be imagined as the "curving of the fabric"
4. For Riemann[41], for any manifold, the metric is Pythagorean[42] in every point in an infinitesimal region – Pythagoras theorem is always valid in an infinitesimal region
5. Example: for the space implicit in ordinary Euclidean geometry, the metric is Euclidean at any scale (in the same coordinate system)
6. Example: the hyperbolic and spherical geometry are just two special cases of manifolds with constant curvature (positive and negative)
7. Many more nontraditional non Euclidean geometries can be generated as special cases of manifolds
8. Key Point: On a sufficiently small scale, the curvature vanishes, that is, the space appears flat - and the metric is Pythagorean

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[38] Otherwise there would be a super-environment-space that is not object-of-study of geometry.

[39] Not so for more modern mathematical and mathematical physics theories, e.g., differential topology or affine differential manifolds have been largely applied in general relativity.

[40] Not so in more modern mathematical physics of space time, where connections are associated to curvature.

[41] Not so, already, in Minkowski space time of special relativity.

[42] We avoid here the most common term Euclidean, to avoid confusion.

9. The space is multi-dimensional – may have any (finite) number of dimensions
10. Riemannian manifolds are "differential" – for they have a topological structure (induced from the metric) that allows meaningful talking about infinitesimals/ limit processes

Cognitive observations about Riemann's curvature

The following characteristics of the Riemannian manifolds make them (our model of) the geometry of the world of our students:

1. The curvature of Riemannian manifold vanishes on "infinitesimal scale" – mirroring the intuitive model we have of both the curvature of physical objects and the curvature of our land
2. The geometry of the figures of a Riemannian manifold is Euclidean on "infinitesimal scale" - this in line with the experience we have of the figures of our land
3. A straight line of a Riemann's manifold is made of Euclidean infinitesimal line elements – this is exactly how it appears a straight line on the terrestrial surface

Riemannian manifolds: non Euclidean only because of curvature

A point of interest is that a Riemannian space is non Euclidean only because of its curvature: if its curvature vanishes (in a finite region or entirely) it is Euclidean (in that region or entirely). This may lead to think that "non Euclidean equals curved" – an easy to fall in misconception, if one is adopting the historical epistemology of non Euclidean geometries. A good counterexample is provided by the general relativity space time: in a region that is flat, however small is the scale, the space time keeps being non Euclidean - because of its metric (Lorentzian).

Riemann and world geometry

We wish to note that Riemann questioned whether physical space should have had the Euclidean metric, suggesting instead it had to be determined by physics, being a feature of the space not independent from the physical objects[43]. Actually, Riemann went much further, for he tried himself to include electromagnetism into geometry. He realized that geometry had to be merged with

[43] This may be understood in the sense that the metric should be established with empirical means (it is not a priori Euclidean – as a Kantian inclination may assert), but also that, being the metric determined by empirical observations, its type (Euclidean or non) depends on the physical laws to which measurements (especially, light rays propagation) obey. Poincaré argued that, being there a convention to be chosen, the physical laws should be modified so to save the Euclidean character of geometry. Einstein went the opposite direction.

physics to achieve a full description of space. Remarkable, is that also Clifford, after having learnt of Riemann's work, tried to include even mass into geometry, as curvature of space (!). But both of them failed, because they sought the integration of fields into curvature of space – instead of space time. Thinking about replacing forces with curvature is not a distinctive feature of Einstein gravitation theory or of his work. As it was shown by Cartan, (1923) and (1924), Newton's gravitation equations can be rewritten without any force in terms of curvature, not of space, but of space time. What is distinctive of Einstein work, it seems to us, and in addition to many other aspects, is having done the "unification" in space time - and having succeeded at it.

The issue of the rigid body

Since in our general epistemological model we have proposed a view of space with general variable curvature, this might appear as inacceptable from the concept/ necessity of "rigid body". The question is much more complex than the issue of the curvature and for it we make reference to Reichenbach, 1977. Accordingly with relativity theory (Misner et al., 1973), here we pinpoint that:

1. special relativity, implies that there cannot be absolute rigid bodies – for the deformation propagation speed is finite
2. in "space" language: you cannot slide a rigid body across a space-tier of space time, simply because there are no such a space-tiers (Lorentzian metric)
3. surfaces of our world do not show any sign of constant curvature (which is generally required to keep the rigid body concept meaningful)
4. the concept of rigid body on a global scale is also at odds with the fact that there is no globally valid reference system (after the dismissal of Newton's absolute space).

3.6 Riemannian manifolds in general relativity

In the following we try to explain how the Riemannian manifold "evolves" into general relativity time space. A conceptual schema of how we see the revolution that general relativity brought about in geometry or the world-geometry is in the successive paragraph. For both paragraphs our general references are: Misner et al., 1973; Noll, 2004; Cartan,1923.

-> This discussion may help understand that: flat space does not mean Euclidean – contrary to the Many Geometries paradigm, as used in common non Euclidean laboratories; the exception does not come from purely mathematical abstraction, but from the very "essence of our physical space".

Before Riemann, space was an invisible spectator of geometry. With Riemann becomes an ever present actor (for example, it bends impossible spherical triangles to close), though not directly observable. With general relativity, "space" becomes a "changing actor" of physics.

We saw that Riemann removes flatness from space: via a metric structure, introduces a curvature of space that, nonetheless, is vanishing on a small scale. What Riemann does not remove, is the assumption that the fabric of space, the metric, is Euclidean (Pythagorean)[44]. Although the Euclidean fabric is distorted on large scale, by the bending and warping of space, the Euclidean fabric shows up intact on a small scale, where the curvature vanishes (the reason for which our Euclidean maps work for small region of the Earth's surface – Earth's surface can be "thought" as a 2D Riemannian manifold).

It is with Einstein that the fabric of space changes, because space itself is jettisoned: space is replaced by intertwined space & time, that comes with its own fabric, non Euclidean (it is called Lorentzian or pseudo-Euclidean)[45]. Einstein is compelled to swap space with space time, because of physical experiments that showed that electromagnetic fields could not possibly be studied with a space independent of time: space and time are intertwined in a Lorentzian fabric (Special Relativity Theory).

-> Therefore, the following two foundational images of the common idea of space are "illusory":

1. space as snapshot in the time flow – where we study figures
2. time as an additional dimension of space – bringing it to four dimensions

The reality is:

1. space and time are entangled in a new type of "environment space" - with non Euclidean metric (Lorentzian), even where is flat

More precisely, Einstein removed a triple assumption that was implicit, more than in Euclid's environment space, in the science of Aristotle of the underlying cathedral (see paragraph, Euclid's geometry deep foundations). Indeed, because of the edifice of knowledge built by the Greeks and

[44] We may use Euclidean, as it is common – if there is no risk of confusion.

[45] It was Minkowski who for the first time provided a complete mathematical treatment of the new space time.

used still by the XVIII century great scientists – mathematical physics assumed, e.g., Newton assumed, that there is a space that can be discussed independently from time (1), that that space is something of absolute (2), and that there is a time that is something of absolute too (3). If space is no longer absolute (there is no "background" on which to "anchor" our thinking), distance too is no longer an invariant. Indeed, it is "distance" in space time that becomes the invariant. Because in space time the "points" are "events", the distance is between couples of events. This distance is not what we would obtain by generalizing Pythagoras theorem from three to four dimensions – as we do at School from two to three, but it is a new "formula" in which distance may take even negative value (pseudo-Euclidean)[46] and figures "have sense".

The "space" so obtained, is four-dimensional, and because its fabric is only pseudo-Euclidean, the space is not strictly speaking Riemannian – it is called pseudo-Riemannian. In it, geometry becomes kinematics, but, theoretically speaking, "Euclidean figures" have no sense (for they belong to non sensical time sections of space time!). Yet, empirically speaking, for "low speed", the fabric of space time is Euclidean.

Einstein made then the second move (General Relativity). To make the kinematics of this space time consistent with the motions observed in presence of gravitational forces, Einstein bends and warps the pseudo-Riemannian space time, gets rid of the idea of force of Newton's gravitation by transposing it into curvature of the pseudo-Riemannian space time and writes the equations of how mass yields the curvature and how the curvature determines motion of mass. The general relativity space time is curved, as Riemannian spaces, and the Riemann's differential geometry can be adjusted to the "pseudo" component.

In a sense, the space time of general relativity has gone one level up of Riemannian manifolds: the space time itself changes in time – the subject of geometrodynamics (Misner et al., 1973, p. 1184).

3.7 General relativity: revolution of geometry

For our purposes and in our view, Einstein's epochal work was the replacement of: <matter, Euclidean space and time, gravity force> with <matter, non-Euclidean space-time>

[46] The physical reason is "loss of simultaneity". One way to assign the "metric" is by: $ds^2 = cdt^2 - dx^2 - dy^2 - dz^2$ (Lorentzian) – which may be positive, negative or null, depending of the type of "event interval".

A pedagogical deconstruction

This replacement may be deconstructed in four stages - so to unearth four component revolutions of the idea of geometry underlying the passage from Euclid to Einstein:

Stage 1. Think of geometry as happening not in "Euclidean space", but in "Euclidean/ Newtonian space time"

Geometry is enlarged to account for the way we experience space, through time and movements - once banished by Euclid, are now taken back and flanked with the explicit concepts of absolute space and time of Newton[47].

Note: our idea to redefine straight lines as a static, kinematic and dynamical model may be seen having here its roots.

Stage 2. Get rid of the force-part of the Newtonian gravity by curving the Newton's space time and thus making it into a non Euclidean[48]

Geometry is further enlarged to account for our experience and model of gravitational force. By curving space time, mass makes geometry a variable (a function of place and of time). The absolute character of space and time is maintained.

Note: this is a key point of our straight line model: straight movements are not those free from force (field), but those "freely falling in a force field" (see also our discussion about how this changes the vestibular neurophysiology model – in chapter 2 and paragraph 4.4). It is the general relativity equivalence principle that is at the basis of this shift. To be precise this is the Newton equivalence principle in the Cartan reformulation of Newton's gravitation (Misner et al. 1977, p. 305, from 9th line onward)

Stage 3. Get rid of the residual Newtonian character of absolute space and time by "distorting" distances of space time and thus bringing in a second non Euclidean component of geometry

Geometry is re-textured by absorbing the special relativity of Einstein's into space time geometry changing it from a stratification of space-slices (Misner et al., 1973, p. 291) to "indissoluble mixing structures" (Lorentz geometry). This second non Euclidean character impact distances, not

[47] Absolute space and time do not show up in Euclid, for his geometry is not about space, time and movement.

[48] Newton- Cartan gravity – a "simple" physical mathematical rewriting of Newton's equations (Cartan, 1923 and 1924), a philosophically still debated situation (see for example, Bain, 2004), pointing to a well open challenge (Hestenes, 2005).

curvature. We cannot discuss Special Relativity, but we believe that one could get a glimpse by simply reflecting on the impact of the relativity of simultaneity on "the commonly supposed snapshot of space we have in front of our eyes": it is an observer dependent space-time slice.

Note: this part we could not possibly transpose it to our students level – they had no knowledge of even a simple Cartesian grid.

The stage is now set: we have come to a variably curved space-time that is made of non Euclidean, but flat micro-tiles. The curvature of space time is mathematically/ theoretically non zero, yet empirically is negligible if the gravitational field is sufficiently weak or if the scale of observation is sufficiently small[49].

Stage 4. Design the geometric equations governing the time-change of space-time geometry and the movement of matter (geometrodynamics)

Geometry fully develops its analytical power to model cause-effect-in-feedback relationships governing movement of mass, curvature of space time and action of mass on space time in relativistic space time. It does so by creating abstract geometric objects (e.g., tensors, forms), living in their own spaces (e.g., dual spaces) and writing equations among them – as in the Einstein Field Equation:

$G = 8 \pi T$

which shows how the stress-energy of matter, T, generates curvature of space time, G, in its neighborhood (Misner et al., 1973, p.42) – the geometrical object G is called Einstein tensor and manages the curvature aspects, whereas, T is another tensor, the one for the stress energy of matter. In this equation is embodied the systemic nature of general relativity, in the words of (Misner et al., 1973, p.5): "Space [time] tells matter how to move" "Matter tells space [time] how to curve"[50]

[49] This is not entirely correct. The discussion should consider sensitivity of the observer to indirect effects of curvature ("we may get the wrong way" because the GPS system did not account for a minimal curvature).

[50] These are infinite-dimensional dynamical systems (partial differential equations)

Seeing straight what looks curving

All the above implies that the movement tracks curving under the effect of the gravity force caused by matter[51] were re-casted as straight movements of a space time curved by matter, with no absolute space or time.

Before trying to get a glimpse of a "curved space time in which there are straight movements that leave curved tracks on a flat space", let reflect on how geometry epistemology has changed, by the absorption of gravity in space time:

1. in our view, it is not a pollution of geometry by physics - quite the contrary: it is freeing our concept of space from another illusory feature (the previous was of being independent of time): that of being independent of what is in it: matter and field. This work is centered on the idea of overcoming the irreconcilability of Euclidean and non Euclidean geometries by extending our concept of geometry into a physics of movements in a curved space. As we have shown, this change of epistemology is not a artifice: it is aligning our idea of geometry to the modern understanding of it brought about by general relativity nearly 100 years ago. This is one of the foundations of our proposal to look at geometry in a "new way": as physics of movements rooted in the new understanding of space (and time).
2. the need for this liberation, by embedding into geometry matter or fields, matured even before Einstein- though it was he that could overcome the barriers to the achievement: "The fact that space-time geometry cannot adequately be considered in isolation from other parts of physics, and hence that its concepts and laws are inextricably interwoven with those of mechanics, electrodynamics, etc., was first recognized by B. Riemann and W.K. Clifford, thereafter by H.Minkowski and A.Einstein, and was particularly emphasised by H. Weyl (Weyl, 1918 and 1949)" (Boi, 2004, p. 432).

Coming back to "reconcile what we see with what general relativity tells us", we provide a few guiding kinematic observations.

The tracks of a baseball and a satellite appear curved. Are these curving trajectories images of the curvature of space time? Yes and no – we could answer. They cannot be, because both the ball and the bullet do go straight in space time – this is a central principle of general relativity. And yet, somewhere, their curving must come out, for we say that "it is the curvature of space time that

[51] Mysteriously - the Newton's "Hypothesis non fingo".

makes them fall". To decipher this conundrum, we think it is necessary to be clear about three key elements – that we have made at the centerpiece of our laboratory design:

1. First: a concept of straight line effective for non Euclidean situations
2. Second: a concept of curvature effective for "making non Euclidean situations intelligible"
3. Third: the concept of distortion of projections from non Euclidean to Euclidean spaces

We note that we have made these three elements pillars in our student's laboratory – in our view, the cognitive issues are very similar.

Straight Line

Straight line is the result of going straight. Going straight, is an observer dependent concept: going south on the straight stretch of state road 51 is:

a. for the driver, going straight: leave a straight trace, I do not turn the wheel, I do not experience any sideway acceleration
b. for the astronaut observing the driver, going curved: an arch on a spherical surface

Curvature

Therefore, since our experience is that of living on a flat surface, we take any meridian to be a straight line. Let now pretend to be at the equator and take place with two bikes on two meridians a couple of meters apart one from the other. We are on two parallel segments. If we head straight, these two parallels will meet at the pole. We have just detected a non Euclidean behavior on an experienced flat surface. Since the tracks "belong" to us flat observers, they stay straight. The curvature is an attribute of the "underlying" on which geometry is acted, an underlying that, of course, we pretend we cannot see from outside (as we cannot see space time from outside) and the geometry of which is external to our perception[52].

Distortion

When we project on a 2D map our meridians, we get curved paths, just check it out on your atlas. Our straight lines were straight but seen on a flat projection look curved.

With these three concepts, we can now take a glimpse of general relativity geometry:

[52] This is the "Riemannian space" of the manifolds.

The space where we live appears and feel absolutely flat (Euclidean). Therefore, it is as if we were on a flat 2D map. Now, the trajectory of the baseball looks curved in our flat "space"[53]. Yet, Einstein told us that geometry is played in space time – not in the space (sky), and, he told us, that if we could live in space time (a four dimensions surface) we would see and feel that the baseball is going as straight as the imaginary cars, on a space time which is curved – as was so the surface for the bikes; so that it is a bit as if the baseball was following a space time spherical meridian and we, by looking at its projection on our sky-map, saw it distorted into a curved line. Therefore: the baseball is moving on a straight-line in space time. Yet space-time is curved. The tracks of the movement is a distorted curve.

Another confusing situation is that of light rays that "curve" while going around the Sun.

-> Light does not curve, it always goes straight[54]; therefore, contrary to many widespread statements, no experiment has shown the "bending of light".

Light rays do not move in "space section", for movement is in space time[55]. And neither helps saying that "because space time is curved then we see them curving according to space time" – if we "could see" space time (as we can see the land and a car going straight on it), we would see them going straight. Instead, the reasoning may go as in the previous example:

-> what "we see" is their tracks which are curved around the sun, for they are distorted projections of rays that are straight on a space time which is curved in a "direction" not accessible to our experience or instruments.

Even more, the "bending of light rays" is erroneously identified as specific of Einstein general relativity. This is not the case. Newton's gravitation, in the Cartan formulation (Cartan, 1923, 1924) asserts the same fact. It is only about the amount of "space time bending", that the two gravitation theories differ [56].

Space time is critical for the unification. As explained by Misner et al., 1973, p. 32, Riemann tried to do step 2 (of the previous deconstruction schema), but he was unsuccessful because he was thinking of geometry as geometry of space not space time. Indeed, a bullet and a baseball shot out

[53] Things here are more subtle, for there is no such a space. Yet, for our purposes is sufficient.

[54] Of course, if we add dimensions to space time, e.g., Kaluza Klein model, straight returns to be a relative concept

[55] It is plenty of articles and books that foster these misunderstandings.

[56] One would wonder what was then proved by the 1919 Sun Eclipse (Eddington). It was proved that the deflection was bigger than that explicable by Newtonian gravity.

of the same place should leave very similar tracks – for they are in the same place of that space which is curved by Earth (in analogy with two points that go straight on a sphere and leave very similar tracks). Yet, the tracks are very different and make the idea appear nonsense. Indeed, when we look at the movements in space time, surprisingly, the lines come to have very much the same curvature: because of difference in speed (!):

-> geometry has "swallowed kinematics" – and let us understand curvature of space time

Therefore, it makes sense to see them as distorted projections on flat space time of straight lines on curved space time.

3.8 V.I. Arnold and R. Thom on formalistic conception of mathematics

In the Box about the general relativity's revolution of geometry, we have seen that geometry is really mathematical physics. On the contrary, in the emergence of non Euclidean geometries, we have argued, there is an assertion of "free from physics". And, as we made it clear, this view of mathematics is unsuitable for our students. It seems appropriate, therefore, to mention that renowned mathematicians made the point for "mathematics of the world", instead of a "mathematics of formal abstractions". We wish to regard this as a convergence point for science and education. Here are some of the sharpest statements.

Mathematical work starts from physical world …

"Mathematics is a part of physics. Physics is an experimental science, a part of natural science. Mathematics is the part of physics where experiments are cheap."

V. I. Arnold, 1997, Extended text of his address at a discussion on the teaching of mathematics in Palais de Découverte in Paris on 7 March 1997.

http://pauli.uni-muenster.de/~munsteg/arnold.html

… to make mathematics purely formal makes it irrelevant to man's world ….

"In the last 30 years, the prestige of mathematics has declined in all countries. I think that mathematicians are partially to be blamed as well (foremost, Hilbert and Bourbaki), the ones who proclaimed that the goal of their science was investigation of all corollaries of arbitrary systems of axioms."

V.I. Arnold, Tribute to Vladimir Arnold by Boris Khesin and Serge Tabachnikov, Coordinating Editors, 1997, Notices of the AMS Volume 59, Number 3

http://www.ams.org/notices/201203/rtx120300378p.pdf

… and irrelevant to man's science ….

« La limite de la vérité n'est pas l'erreur, c'est l'insignifiance »

R. Thom, Reported by B. Tessier in B. Teissier, GEOMETRIE ET COGNITION : L'EXEMPLE DU CONTINU, CERISY SEPTEMBRE 2006

http://arxiv.org/pdf/0806.3864v1.pdf

The trend has changed ……

"[…] the divorce between pure mathematics and the sciences created by the (I would say criminal) formalization of mathematics and of mathematical education. The axiomatical-deductive Hilbert-Bourbaki style of exposition of mathematics, dominant in the first half of this century, is now fortunately giving place to the unifying trends of the Poincaré style geometrical mathematics, combining deep theoretical insight with real-world applications"

V.I. Arnold, An Interview with Vladimir Arnold by S. H. Lui, NOTICES OF THE AMS VOLUME 44, NUMBER 4

http://www.math.upenn.edu/Arnold/Arnold-interview1997.pdf

3.9 Is our Universe Euclidean?

In this work we have proposed a view of geometry blurring any contraposition of Euclidean and non Euclidean and combining them as coexisting aspects of our world and already discussed the implications of general relativity on the curvature of space time and of space time fabric. In this paragraph we will collect these implications to discuss (our) view about the question, "Is our universe Euclidean?".

The limitations

We will develop our discussion in the most basic way: ignoring contemporary physics (string theory) and inflationary "great visions" and sticking on a fairly established and shared model: that of general relativity. We are also aware that the whole debate may be taken to a philosophical, yet not speculative level (if we stayed, for example, at the level of H. Poincaré with his conventionalism), but we will stay at a much less sophisticated level. It should also be clear that we

talk about models. For example, when we talk about the non-Euclidean character of the surface of the Earth, we actually mean the surface of the sphere or the geoid by which we model the Earth, not the real physical Earth which appears to be a continuous body on our scale, but morphs into matter/energy fields as soon as we get onto microscopic scale.

Rationale

We do it, because we believe it important to "relativize" some common opinions, expressed in particular in the educational sector, for example, in Clark, 2012, p. xix, about a supposedly "scientific proof that Euclidean geometry accurately models the theoretically visible universe" and that "this data finally demonstrated that space has curvature zero within the visible universe".

Our interpretation

In our understanding, these opinions assume, explicitly or implicitly, the many-geometries paradigm. In so doing they "miss the point": our universe is both Euclidean and non Euclidean (Riemannian) and it is not a question of "measuring" – it is theoretically so, by general relativity (which has been in turn subject to experimental control, e.g.: the GPS, the precession of perihelion of Mercury, the light rays). In our discussion we are going to use also the epistemological analysis of chapter 4.

Indeed, in Clark, 2012, the non Euclidean geometries are cited as one of the dilemmas that "conspired to phase Euclidean geometry out of the school curriculum". In Clark, 2012, p. xvii, we have

"A third pedagogical dilemma emerged from […] the discoveries of non-Euclidean geometries […] Hyperbolic or […] Elliptic [Spherical]. But the non-Euclidean revolution did not address the question as to which, if any, these postulates [of Hyperbolic or Elliptic geometry] is true in physical space. It only told us that [… is…] a question of empirical science. […] During the eighties and nineties this question suddenly came to the forefront. […] The question as to whether space was elliptic (positive curvature), hyperbolic (negative curvature) or Euclidean (zero curvature) was directly tied to the question of […] cosmic expansion […]. We lived each day with the prospect of reading headlines announcing that space was now determined to be elliptical, hyperbolic or Euclidean. In this context Euclidean geometry in university courses is often truncated to make room for non-Euclidean unit." Clark, 2012, p. xix, eventually asserts "This data [of the WMAP probe launched by NASA in 2001] finally demonstrated that shape has curvature zero within the visible

universe. […] As a result we at last have scientific proof that Euclidean geometry accurately models the theoretically visible universe."

Our position in brief

This view of geometry corresponds to the many-geometries paradigm. On the contrary, in this work we have argued for a more modern paradigm (Riemannian and general relativity) by which: the space has no meaning; the space time is neither elliptic nor hyperbolic, for its curvature is variable; locally, space time is Lorentzian and, under some conditions, Euclidean; space time cannot be dissociated from matter/ energy. In our opinion, these fundamental aspects of our universe cannot be comprehended without the non Euclidean "concepts" and a radically different idea of geometry (that we have tried to explain in this work).

On the other side, as we argued, Euclid's geometry relevance for any student cannot be obscured by (non Euclidean) geometries that were by act of birth purely formal and that were built on no other foundation than the Euclid's system (!). From the perspective of this work, it is hard to see how they could represent a threat or a pedagogical dilemma conspiring to phase Euclidean geometry out of our curriculum. Even if we restricted to the value of Euclid's work as a model of hypothetical-deductive system – as we have argued, non Euclidean geometries "build on it".

It is important to note that the cosmic expansion debate (the WMAP probe was launched with that research aim) does not strictly concern the space time geometry – which we already know to be curved and have non Euclidean fabric by general relativity. It concerns something which could not be farther away from our world: the overall shape of the overall universe. It is a point of view antithetical to the "local point of view" on which relativity, Riemannian geometry and this work are built. In that setting, the assumption that the mass/ energy distribution be uniform is made – which makes no sense if we want to investigate the geometry of space time, for it is not uniform at all.

We revise in some details our arguments.

Detailed arguments

At detailed level we just recall that (see paragraphs about general relativity and references to Misner et al., 1973:

1. ordinary common conception of three-dimensional space as such has no meaning in any discussion about "space" – it is space and time that has sense.
2. that a flat "space" implies its "being Euclidean" is false – as indeed it is the case for our space time: it is everywhere Lorentzian, that is, non Euclidean (non Pythagorean metric)

3. space time is neither Pythagorean nor flat – this general relativity: it would be so only if there were no energy/ mass in the universe
4. not only space time is curved and its fabric is non Euclidean, but its curvature is even variable from "place" to "place"
5. not only is variably curved – it is even changing in time his curvature (geometrodynamics)
6. any statement of the type "physical measurements having confirmed that space or space time is flat or Euclidean" appears incorrect when the distinction between "empirically versus theoretically flat" is not made (see, the epistemological challenge in Attachment 4)

A broader view

Clarified our position that non Euclidean behaviors are a vital means to understand and discuss the true space that is around us, we wish to relativize the value of our position too.

In a broader sense, we believe, there is no definitive answer to these questions – they might even become irrelevant or non sensical with future development of science. A glance at the changes of physics understanding of the universe in the last 100 years shows how futile is to hold on to seeking absolute and necessary (the Greeks aspirations) truths in the domain of physics[57]. There are deep reasons to take such a stance. Further to the classic works of Feyerabend and Kuhn, that bring science back into a socially driven project – and therefore a priori bounded, we believe the critique of Rorty (1979) to be still "revealing". He shows that, by using the same enquiring mind that we use in science (by science we always mean mathematical physics) to question the object of science, the unescapable conclusion is that there is no such an object. Far from being a skeptic or a defeatist attitude, his is a therapeutic call, to help us free our work from illusions.

[57] For example, a new gauge theory of gravity on flat space-time has recently been developed by Lasenby, Doran, and Gull - see Doran (1998).

Chapter 4: The epistemological model

This chapter builds on the previous chapter analysis to answer the second research question:

2. What is a specific epistemology of geometry (Euclidean & non Euclidean), and a didactical transposition, built on the key ideas of Riemannian manifolds and general relativity, that can fulfill the psycho-educational requirements?

According to Chevallard, at any one moment, in our society

"[there are some bodies of knowledge] tacitly regarded as unteachable" and "there are always somewhere in society some people striving to secure teachability for some previously untaught bodies of knowledge" (Chevallard, 2007).

In chapter 1 we have argued that common didactical transpositions of traditional non Euclidean geometries are ineffective to our students – and in chapter 2 we have drilled into the issue from a psycho-educational perspective. In chapter 3 we have argued that these geometries body of knowledge is by "act-of-birth" ill –disposed for an effective transposition. Therefore, in this chapter we seek a solution by changing the body of knowledge. Our new body of knowledge is assembled from key elements of Riemannian manifolds and general relativity and we redesign the epistemology of geometry based on these elements and, from there, we design our transposition.

4.1 The idea

The student world

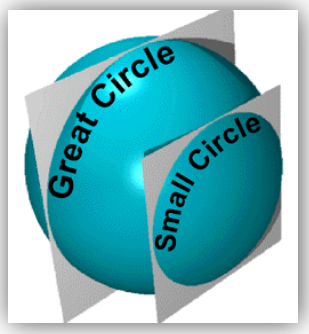
Our psycho-educational requirements for effective learning require that students experience a (non Euclidean) physics of movements with "body and soul" (in particular, "embodied" in the larger sense hinted to in chapter 2).

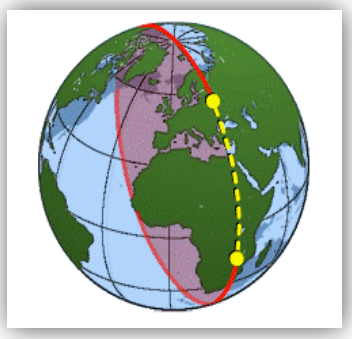
We believe that the geometry the students experience is that of the 2D land - space is experienced by birds, not ordinary students.

That land, netted of the landscape, appears undeniably flat and its geometry, from the A4 paper sheet to the horizon, is experienced as the Euclidean geometry of a bounded region.

Students know, because they have been told, that their flat land is actually spherical – but of it, they have no experience.

Issues with common approaches





Figures 4. 8

In common approaches to non Euclidean geometry, the Earth surface's geometry is regarded as non Euclidean of spherical type[58]. The student is led into thinking that the geometry of his experience is only appearing Euclidean, but actually it is not – appearance is just an effect of the measurements errors and perceptions. The reality is that it is spherical. He would be told, and shown on artifacts, that his straight lines were not actually straight – were curving lines.

-> Accordingly to our psycho-educational analysis, he would therefore "repudiate" those (no longer) straight lines of the land and, whatever would be said about their behaviors, it would have no impact on the student concept of straight lines (for he would feel them as curving lines of a sphere!).

In all other laboratories in which the Earth is not the principal concern and its place is taken by the spherical artifact (or any other), the psycho-educational issue worsens:

-> it is not "the straight lines of the student that are under investigation", but "those of an artifact".

Therefore,

-> for the student, there is nothing of non Euclidean in "the geometry of the great circles of a spherical surface", since "they are not his straight lines", but just "curving lines on a curved object". There cannot be any surprise with their not behaving as his straight lines of his plane.

[58] More precisely, in the Many Geometries paradigm: the mathematical physics "model of the shape of the Earth" is a "model of the theory" of the non Euclidean spherical geometry.

The proposed approach

We believe that the core issue is that of "having many geometries", e.g., of the plane, of the spherical surface - for it immediately leads the student to: "a straight line is a true straight line only if it is on a plane" - for our students do not classify straight lines according to their mathematical validity or mathematical classification[59], but only by meaning/ truth in their world[60]. Shortly: if there are many geometries, all those that do not correspond to his experience are meaningless for the student.

Consequently, in this work, we develop a different approach. First, we lead the student to confirm the straight lines of his land as "legitimate", but on a very different basis. We lead him to developing his cognitive awareness of the broad and deep perceptions and cognition associated with "his going straight movements" – with an outdoor work, body centered, sensorimotorial, etc. From this extended geometry cognition, we lead him to look at straight lines (and triangles, etc.) in terms of "how they behave in his very world", instead of in terms of static figures. The student will observe that: on a small scale, straight lines, triangles, etc., behave as ordinarily; on a larger scale, on Earth surface, new puzzling behaviors appear. Static geometry is abandoned for a physics of movements – really "embodied".

-> Non Euclidean or Euclidean are types of behavior – not an attribute of alternative for geometries. Geometry becomes a set of behaviors. On small scale, the behaviors are Euclidean and geometry will be said Euclidean. On large scale, the behaviors are non Euclidean and the geometry will be said non Euclidean. And being scale observer-laden, what is Euclidean may become non Euclidean to a different observer – and vice versa.

Therefore, on the overall land, the geometry is both Euclidean and non Euclidean (and it will be just a Riemannian manifold geometry). The concept of curvature will arise as a natural way to interpret different behaviors and will be grasped only by isolating the concept of space (in our case surface) as the "underlying to the geometry of figures" – that we discussed in chapter 3.

As it will be explained in the laboratory design, we think of a didactical progression by "stages" with built-in paradoxical situations that, puzzling the student's perception and emotions (empathic situations), would boost him to restructure his images, intuitive models and perception schemas – in

[59] See also our discussion of Van Hiele versus Piaget classification and structures in chapter 2.

[60] Our student is anchored to meaning and truth (world), not to validity (mathematics).

line with the two fundamental characters of "staged" and "cognitive rupture" of effective learning in mathematics expressed in the chapter about the psycho-educational requirements.

In the following we start from "redefining geometry" and then "we redefine the straight line model". First step is revising the Many Geometries paradigm, deconstruct it and construct the epistemology we need.

4.2 Analyzing the Many Geometries paradigm

The paradigm

Non Euclidean geometry laboratories and mainstream monographs[61] adhere to the paradigm of "many geometries", i.e.: there are

1. several incompatible geometry systems, that correspond to
2. several incompatible choices of "space" (Riemannian manifolds), that correspond to
3. several incompatible 2D surface models

In addition, the following view is common:

4. the problem of world geometry is that of determining what geometry best fit into experimental data[62]. (Greenberg, 1994, p. 45)

Euclidean experience[63]

Since, we use the expression that "our students experience the world as Euclidean", we clarify what we mean. We do not intend that their experience is experimentally incompatible with non Euclidean geometries. The perceptive data are compatible with both geometries, because it is enough to make

[61] We could not find any work for middle school that radically departs from this epistemology. A short check at the high school level produced the same result. Even at tertiary level we have found this epistemology as common – beside other more advanced and off track for us, e.g., Klein group theory, projective geometry and complex analysis.

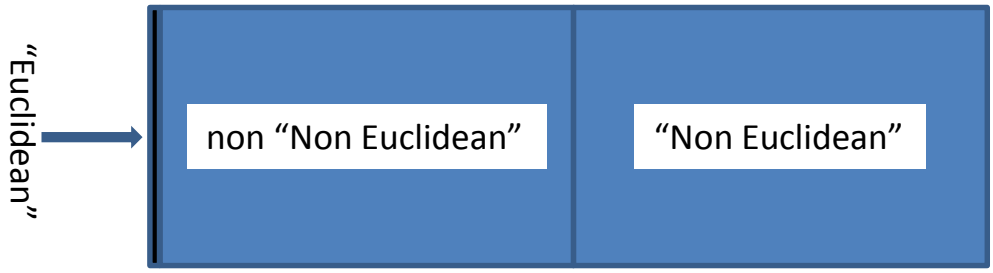
[62] And, consequently, debates (still going on) about whether the geometry of the world is Euclidean or non Euclidean (see in chapter 3, Is our universe Euclidean?)

[63] This topic is discussed as one of the challenges met in this work – in Attachment 4.

the hypothesis that non Euclidean deviations are so small that we cannot perceive them. Instead, we mean that

-> the students have no direct and egocentric experience of what is “distinctive” of non Euclidean geometries – e.g., that parallels change their separation

A more precise but convoluted characterization would be that of saying that their experience is non “non Euclidean”. In the symbolic schema below, we have tried to convey the point that: any measurement of finite precision either tells us that the world is non Euclidean (and then we are finished) or tells us that the non Euclidean character has not shown up (and then we go on), but in no way can tell us that the Euclidean character has shown up – this requires infinite precision (the breathless segment on the left)



The paradigm lineage and its issues

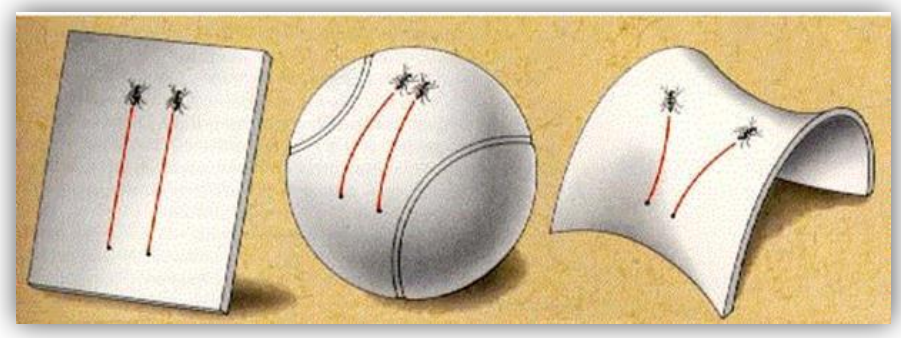
In the paragraph, Non Euclidean geometries as per their origins, of chapter 3, we have provided background information and analysis. Now we return on some key psycho-educational issues.

While Euclid’s geometry appears as knowledge of the geometry of our physical experience of the world (shortly: of the world as we experience it), the non Euclidean geometries “by their act of birth, have no correspondence with the world of our direct experience”.

At the time of their birth, the possibility that the world’s geometry could be non Euclidean opened up. Indeed, it was not possible to exclude that non Euclidean facts would appear by improving the accuracy of measures. For example, in measuring the sum of the internal angles of a triangle (Greenberg, 1997, p.291). If one thinks by the Many Geometries paradigm, the question of “which” geometry is “true” for the universe is still open. Yet:

-> With respect to our psycho-educational requirements, the possibility that “the geometry of the cosmos of theoretical physics be non Euclidean” is irrelevant to our student cognition of geometry.

Though non Euclidean geometries are abstract, there was found a way to “avatarise” them in our world. Indeed, the two most well-known non Euclidean geometries, the hyperbolic and the spherical, can be “modelled” on surfaces such as, respectively, a saddle and a sphere.



“Modelled” does not mean that these surfaces represent the corresponding non Euclidean geometries. It means that, if we impose the name of “straight lines” to some specific curving lines of these surfaces (great circles in the case of the sphere), then these “straight lines” behave not as Euclid’s straight lines, but accordingly to the rules of the straight lines of abstract hyperbolic and spherical geometries.

-> With respect to our psycho-educational requirements, this model cannot possibly be acceptable. In particular, its straight lines do not “have sense” to our students. Our students “would not understand why they should be surprised that, “lines that straight are not”, do not behave as straight lines[64].

The surface models have constant curvature[65], respectively, negative and positive. In fact, the hyperbolic and spherical geometries are characterized as having, respectively, curvature -1 and + 1.

The Many Geometries paradigm imposed its “epistemological glasses” on Riemannian manifolds, when classified them as a general abstract geometries of multidimensional curved spaces of which the abstract non Euclidean geometries are particular cases.

-> With respect to our psycho-educational requirements, this epistemology of Riemannian manifolds is even more irrelevant to our students.

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[64] Without adding that they are even on a curved surface.

[65] Gaussian curvature.

In the following we deconstruct the paradigm and shift to our proposed view.

4.3 Shifting to One Geometry: Euclidean and non Euclidean

Deconstruction and reconstruction

We may deconstruct the previous paradigm as follows:

1. Initial Paradigms:
   a. One Geometry: Euclid's
   b. Geometry-of-Figures
2. Discovery of non Euclidean geometries
   Shift: from "One Geometry: Euclid's" to "Many Geometries"
   Many mathematical geometries and one physical geometry (to be determined by physical experiments, either Euclidean or non Euclidean[66])
3. Creation of Riemannian Manifolds
   Shift: from Geometry-of-Figures to Geometry-of-Space + Curvature
   (see the paragraph, Riemannian manifolds in chapter 3)
4. Assimilation of the Geometry-of-Space + Curvature paradigm into the Many Geometries:
   "Many-Geometries map into Many-Spaces" – each non Euclidean geometry corresponds to a particular case of space and of curvature

Now, we can reconstruct geometry as we need it. Instead of the assimilation step, we propose to turn things the other way:

5. Assimilation of the Many Geometries paradigm into the Geometry of Space:
   One-Space is built combining many-geometries (many geometries go into one space)
6. Resulting shift leads to: One-Geometry: Euclidean and non Euclidean
   <u>In one geometry of a curved space several geometries of figures are possible</u>[67]

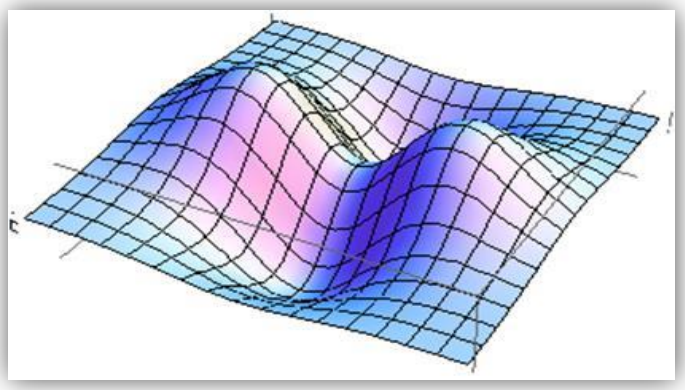
[66] But not both, since mathematical geometries « are» exclusive alternatives.

[67] Our laboratory on the land will have one only "geometry of figures" (the spherical).

This view simply corresponds to a different way to look at Riemannian manifolds, which we are going to "illustrate" by ordinary surfaces.

Picture of One Geometry

We can think about a surface that changes curvature from a saddle to hemispherical, to a plane. In different regions, we have different geometries – "mutually incompatible geometries". The "old" geometries become of "bounded validity" – they are regional.



In our conceptualization of Riemannian manifolds, geometry is on two levels: there is the "geometry of space" (the surface) and the geometry of "figures of space" (figures on the surface). We can "regionalize the figures behaviors" as we wish, by assigning the geometry of space in the appropriate way (i.e., by the appropriate metric, see Berger, 2003, p. 174).

Non Euclidean and Euclidean

What makes this surface capable of representing Euclidean and non Euclidean behaviors is the curvature - without curvature it would be flat and, therefore, Euclidean[68.] The change of curvature from a region to another provides a means to have different geometries (of figures) in different places.

[68] Contrary to the Many Geometries paradigm– this is not true in general. A space may be flat and non Euclidean – if the metric is not Pythagorean (see chapter 3, on general relativity).

Yet curvature, as we experience it in the world, and as we need it for the student to work on the land, is much more: the more we increase our resolution of the surface, the less is the curvature. The most striking example is our land.

This essential character of our experience of curvature we find it beautifully mathematized in Riemannian manifolds geometry – allowing to have two geometries (one always Euclidean) in the same place – as we explain.

Locally Euclidean and student centered

On a sufficiently small scale, the curvature vanishes, that is, the space appears flat and the figures behaviors are Euclidean[69]. That means: the space and behaviors may be non Euclidean, but on a sufficiently small scale all behaviors become Euclidean. It is "the infinitesimal structure of curvature" (Berger, 2003, p. 172), that provides a means to have two geometries (of figures) in one place - by simply changing scale.

-> With respect to our psycho-educational requirements, this is a pillar: it legitimizes[70] the Euclidean experience that our students have of the world as well as all their local Euclidean geometry, which, yet, it will not be true on larger scale. Most critically: it is exactly by abandoning the global perspective of geometry, in favor of the local, that we match how our students experience and think of the world: egocentrically and on their own scale.

This perspective is essential to both Riemannian manifolds and general relativity (Misner et al., 1977, chapter 1): we can build a picture of the global geometry and of the universe by local

[69] This character cannot be captured without "differential geometry" – that is with traditional non Euclidean geometries, and consequently, even in the case of a spherical surface, the epistemology of the Riemannian manifold description is much richer - mathematically and didactically.

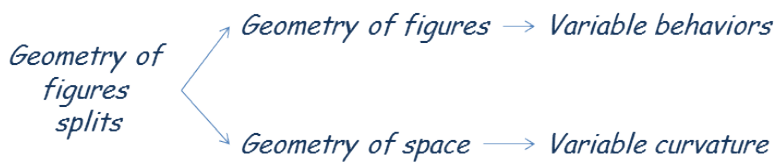
[70] In a socio-constructivist view it is the other way around: it is our experience that renders the construction of Riemannian manifolds a meaningful endeavor.

observation and analysis – in contrast with the many geometries that adopt a global point of view, and in line with a generalized view of embodiment.

Space, behaviors and geometry

Geometry as a "variable - An important outcome is that the geometry becomes a "variable"[71]: geometry of figures[72] changes depending on the place and on the scale. Consequently, figures and their properties change from place to place: an equilateral triangle of a flat region has angles of 60°. Not so when it is on the hemispherical region – it may have all angles of 90°. Two straight lines that on small scale have no changes in separation may converge because of the curvature.

-> Our students will observe changing behaviors only by change of scale (not by change of place) – because the model of the land that we use (a sphere) has constant curvature. We could have used a variable curvature to represent landscape resulting in behaviors changing by change of location. Indeed, the advanced students worked also on surface models with flat and curved regions, where they observed behaviors changing "horizontally".



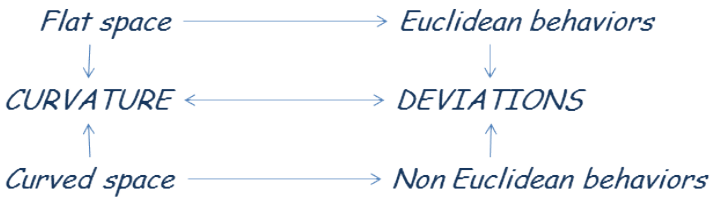
Properties become behaviors – Because geometry becomes a variable, we have been speaking of behaviors of figures – meaning that figures existence and properties change depending on where and on which scale they are observed.

This is exactly what happens in general relativity, where the change in separation of parallel straight lines (in space time) is studied to relate it to curvature/ gravitational field: the higher the variation, the "higher" the curvature/gravitational field of space time. Non Euclidean "geometries" translate into the variety of deviations from non Euclidean behaviors that may appear by increasing the scale

[71] We could say a "function" of "regions of space". We prefer "variable", because transfers the idea that it is something that we observe changing while moving within the space.

[72] Geometry of space is not a variable, if we think of it as, in Riemannian manifolds, the entire metric tensorial field. Yet, in our world, this is not entirely true. In geometrodynamics the entire metric tensorial field changes in "time" – it is in a sense a time-function, and becomes itself a "variable".

of observation or changing place or regions[73]. Curvature is simply a means to account, from within the space, for such deviations. Everything is centered on behaviors – if we wish, curvature and "non Euclidean" are just label for "sets of behaviors".



The concept of behavior is a central pillar in the design of our laboratory.

-> With respect to our psycho-educational requirements, the "behavior based geometry" puts the observer-student in the center and that geometry of figures as the "resulting pattern of observation"

-> The student develops a cognitive approach to "understanding the world" much more effective in dealing with world dis-homogeneity and diversity – for this epistemology of geometry can accommodate a wider variety of experienced geometrical behaviors

-> The student cognitive processing is much more flexible – by developing a paradoxical logico-linguistic intelligence or verbal expression (e.g.: straight and curving line; flat and curved) to account for concrete kinesthetic and visual experiences (as they will be engineered in the laboratory) led by a paradigm of co-existence of (apparently) opposites.

Short summary

1. Traditionally incompatible Euclidean and non Euclidean geometries become geometries that may coexist in different places and/ or on different scale on the same space (Riemannian manifold). One single 2D surface model may combine different non Euclidean geometries.

2. If the space is Riemannian, non Euclidean behaviors are all explained as effects of curvature; on small scale the space is everywhere flat, therefore the geometry is therein Euclidean.

3. Our land model is a two-dimensional Riemannian space: wherever we go, we experience flat space and Euclidean geometry; yet, on larger scale geometry appears non Euclidean (of "spherical type").

[73] As explained in the chapter 3, because of our objectives, in general, we equate flat manifold with Euclidean – which is not always the case, as, for example, in Lorentz space time.

4.4 The straight line model

We have redefined space and geometry as we need them. Now we redefine "what is in space" (the figures) - as needed by our students. The idea is simple:

Space as "place of movements"

In Riemannian manifolds space is an explicitly recognized actor of geometry, which is "observed" from within. How is this possible? The answer comes from general relativity: it is the movements (world lines) that "give us the form", that "reveal" the structure of space time (the haystack picture in Misner et al., 1973, p. 6). We definitively abandon the traditional epistemology of geometry as mathematics divorced from physics (see Arnold's comments in the last paragraph of chapter 3), to embrace that of geometry as physics of movements – in general relativity settings (curved spaces kinematics). We will not work in space time but in a two dimensional "pure" space (surface). Therefore:

1. it is the straight movements[74] that "will probe" space and

2. it is through the observation of their deviations from Euclidean behaviors that we get to know the space curvature.

-> Consequently, we only need a model of straight line – no more many geometries, many straight line models.

Requirements for the model

From our psycho-educational requirements analysis we know that

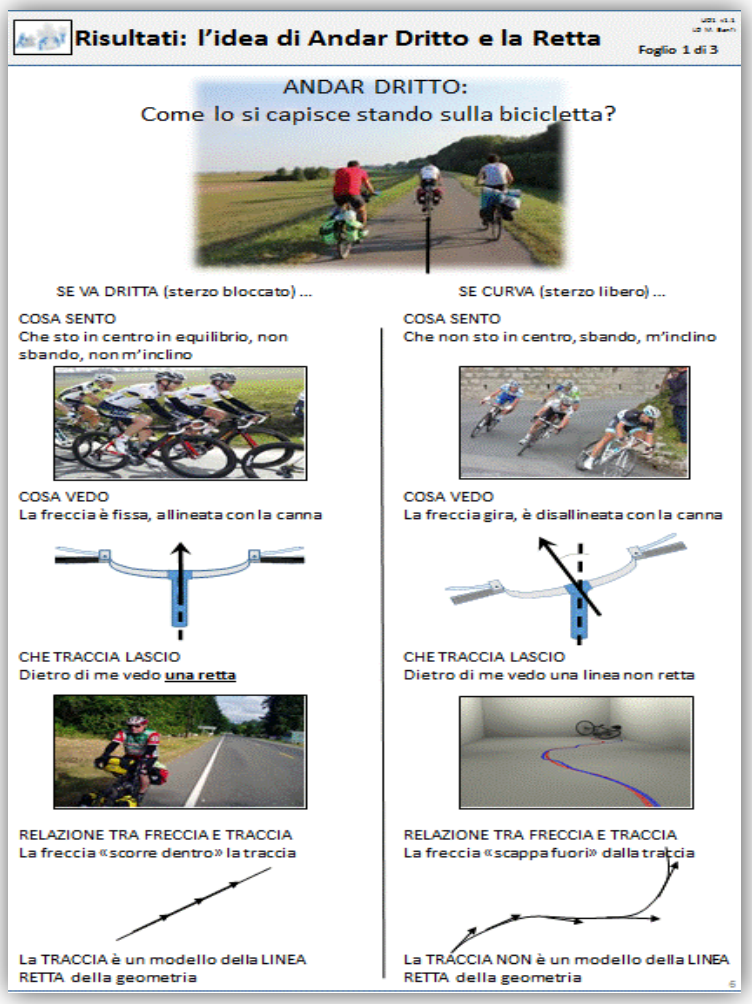
-> The student idea of straight line is rooted in his sensorimotorial cognition.

We have to make the idea work in both Euclidean and non Euclidean situations, that is, both on flat and curved surfaces, and we have to prevent the student from reorganizing cognitively his

[74] We use interchangeably straight movements and straight lines.

perceptions, so to discard the straight lines of the curved surface of the Earth in favor of "real" straight lines beaming off in space[75]

The model



The model is reflected in the handouts of Didactical Unit 1 – in Attachment 1. The model is first of all egocentric and local:

Egocentric - the model is centered on the physical body of the observer (the student) moving along the straight line; the embodiment is of a geometry the epistemology of which is mathematical

[75] As we are going to see, our solution is that of combining a specific model of straight line with two (semantically) distinct models of curvature.

physics of movements in curved spaces – differently from the literature known to us, in which "geometry is seen as mathematics" (see chapter 2).

Local - the observer's perceptions cover a region sufficiently small to be perceived as flat

The model is made of three parts: dynamical, kinematic and static-geometry

1. Dynamical sensorimotorial perception[76]: while the student is going straight, he does not feel any state of stress[77] transversal to the direction of the movement. Equivalently, but easier for children: we do no careen or bend either left or right.

Our concept corresponds to the general relativity concept of a straight line as a path of a "freely moving particle" – a manifestation of the principle of equivalence of general relativity. Before discussing this essential and critical epistemological point, we note that, technically, it is a Newton-Cartan equivalence principle, because we are concerned with (curved) space and time of Newton (Misner et al., 1973, p. 365) – not space and time of Lorentz.

Epistemologically, our concept is critically different from that in which a particle is moving

1. "free from forces" or 2. "not accelerating" or 3. "with constant velocity" or 4."inertial".

that are common in classical physics and in differential geometry - for example, in Berger, 2003, p.35: "The kinematic interpretation of geodesics [straight lines on a curved surface or Riemannian manifolds] is that they are the trajectories followed by a point moving on the surface with no force applied to it"; and in psychology and cognitive sciences - for example, in the "La droite et autres lieux" in the complex model of Teissier (2006), or in Longo & Viaroug (2010), p.13.

We explain why we cannot use these definitions. In our work we are not confronted with the customary Euclidean view of the world, but with how "to intrude" in it non Euclidean behaviors. If we think in terms of "free from forces" or any other of the four above, we end up with movements in flat space, with no curvature and no non Euclidean geometry. The reason is in the core of general relativity geometro-physical structure.

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[76] We underline that perception concerns a (micro) span of time and "variations of values" – not the theoretical one instant of time and values. The word "apperception" is sometimes used to refer to this aspect, but we avoided it because it is not a "standard".

[77] We avoid using the word « force » for many reasons. One is that our perspective is centered on the student's body - the feelings of which can be modelled as stress tensor field – not external punctual arrow shaped forces.

To get the gist of it, we start by observing that if a body movement is acceleration free or free from forces, there is no mass in the universe, there is no curvature of space and the space is flat. Therefore, any definition of straight line that uses one or several of the previous four, is either "empty" or implies by definition Euclidean space.

Let now consider a body moving or still in the International Space Station: it is freely floating inside the station. Is there absence of gravity? No: gravity tends to zero at infinite distance. Is it acceleration free? Neither. It is actually falling into the curved space time following a straight line. It is goings straight. We have a perfect straight line of a curved space (see, Misner et al. 1973, p. 4 and 5). The essential ingredient was gravitation – force and acceleration: "weightlessness" follows from acceleration – not from "free from forces". Let now consider two falling bodies from a tower. They also fall side by side on straight lines, parallel, but the separation of these parallel straight lines decreases, though imperceptibly (roughly speaking: the gravitational field is central). We have now two parallel straight lines, of bodies subject to forces and, yet, perfectly local inertial frames, that show non Euclidean behaviors. Tidal gravitational effects are the core of general relativity non Euclidean geometry of space time[78].

-> To recap. It is exactly because there is the force of gravity, the gravitational acceleration and the non constant velocity, that "the space" can be curved, that we can start talking of straight lines in curved space and that we can speak of non Euclidean behaviors. It is so, that we arrive at "one definition only of straight line valid for both Euclidian and non Euclidian space or spaces" - that reflects the real going straight movements (worldlines) – instead of the absolute ideal straight lines of absolute space.

As a further example, a basketball shot up and falling along a parabola to the ground (in absence of atmosphere) under the Newton's force of gravity is, for general relativity, a freely moving particle: it follows a straight line of space time.

We do not see a conflict with the cited neurophysiology and cognition points of view – even in their specific domain of research. Indeed, it seems to us, that the information processed from the vestibular apparatus of a parachutist in free fall would keep rendering an image of straight line – in line with our model.

[78] The fact that we do not require Einstein's special relativity – Newton Cartan is enough, make them looking interesting, in our opinion, for a laboratory with students that have some basic understanding of Newton's gravitation (Lorentz space time requires an advanced understanding of mathematical physics).

Returning to the very pragmatic point of view of our students:

-> We believe that the model we propose is already well embodied in our student perception

The following kinematical and static parts of the model are actually very entangled and it is for sake of communication that we decided to present them separately. In differential geometry "velocity" is ubiquitous. The major part of modern textbooks do not expel kinematics from geometry – actually is the opposite; for example: Berger (2003) and Sternberg (2012),. Yet, from a tradition codified in the first books of Euclid, elementary geometry is often presented as "static". So we decided to discuss them separately.

2. Kinematical sensorimotorial perception: when the student is going straight, observing the trace a bit behind him and anticipating its image a bit ahead of him, he sees that his direction imagined by an arrow "flows within" the trace. From a mathematical point of view, this corresponds to the concept of straight lines as auto parallel transport (Sternberg, 2012).

From a neuropsychologist point of view, in our understanding, the persistence of vision and anticipation of vision corresponds to the "saccadic eye" (see chapter 2).

3. Static visual image: this corresponds to the customary way of defining straight lines in the plane in elementary geometry – as static shapes: a line which is bent neither left not right, that keeps in the middle. We may link it to the above by saying: an arrow flowing through the line that does not turn. From a physical mathematics point of view, this corresponds to the concept of straight line for flat space - velocity vector of fixed direction. From a mathematical point of view, this corresponds to the concept of straight line for flat space - tangent vector of fixed direction. Note that if the space is not flat, it is the notion of parallel transport that is required and we turn back to the kinematical sensorimotorial.

A comment on the minimal path definition

The path of minimal distance is a common definition for straight lines. From our point of view, this is a property of straight lines - a metric property, that requires a theory of measurement and of instruments of measurement (see Reichenbach, 1977). It does not concern meaning (semantic) – in particular, for our students. Furthermore, in differentiable manifolds, the non Euclidean behaviors of straight lines do not emerge from metric considerations: in particular, in general relativity, they stem at the level of differential topology and connection manifolds (Misner et al., 1973). Even if we

restrict to metric manifolds and their considerations, there may be straight lines that are not minimal paths (helices on cylinders, some arches on the sphere, etc.) and situations where minimal paths do not even exist (e.g. paths around a piercing on a sphere).

From our student centered sensorimotorial point of view it is an "abstraction": "going straight" is about "acting, feeling, seeing, sensing, thinking" – and it is already inherent in our students, it is not about a mathematical property of geometrical object (a path) to be used to define abstract straight lines.

4.5 Two curvatures

As we explained in the Box, Riemannian manifolds, we distinguish semantically the curvature of the "figures" from the "curvature" of the environment space in which the figures live[79]. On this distinction, we have built our strategy to cope with the central perceptive issue of the student, to which we have hinted in the previous paragraph about the straight line requirements:

-> issue: in front of straight lines of a spherical surface the student sees curving lines, because his perception "adds up the curvature of the surface to the straight line"

-> strategy: to keep the two epistemologically radically distinct curvatures, well distinct in the laboratory and train the students perception to discern "what is done by the figure" from "what is done by the space to the figures"

-> one of our students correctly said, that a straight line on the spherical surface was indeed a straight line because "it was not it that curved, but the place where it was "running on"". A second said that, she "could not see anything strange in its being straight [of a straight line running across some hills] because [the bike tracing] the straight line would not bother about the inclination [curvature] of the hills"

We believe that failing to recognize this point or to think of the curvature of the space as something which "adds up" to the curvature of the lines (for example, in Arzarello et al., 2012 p. 94) makes much harder for our students to really see "straight lines" on curved surfaces. This point of view is very different from that which is common in non Euclidean geometry. For example: "[…] make it

[79] Nearly always, we use the "ing-form" for the curvature of the figures – as in "a curving line", and the "ed-form" for that of space - as in "a curved surface". In our intentions the ing-form would give the idea of an "open process", as it is for the movements that make up the figures, whereas the ed-form would give the idea of static attribute.

clear that straight lines of the plane do not exist on the sphere" (Lénárt, 2013, p.15). In our laboratory,

-> the student learns that "what makes a straight line a straight line" is something in itself: there is no difference between straight lines of the plane and of a sphere, for the difference concerns "the underlying on which geometry is played" (the plane of the sphere)

Missing this central point, in our opinion, is the root of a widespread misconception: that in general relativity light rays are bent by the Sun. Indeed, the light does not bend at all: it follows perfect geodesics in space time. It is the space time that is bent.

The point is central also for the circumference; we consistently believe that:

-> Failing to distinguish the two curvatures, leads to confusion about what is a "circumference"

Our students did not fail to recognize that on their land not all circumferences "have to turn" (are curv-ing lines). There are circumferences that are straight, e.g., a circumference centered on the north pole, when the radius reaches one quarter of the circumference of the Earth, becomes a straight line: the equator. Circumferences may be straight on the spherical surface, because it is the space that takes "charge" of keeping the distance from their center constant. This is very different from defining straight lines as the great circles.

-> Our students discover that on the land and on the sphere, circumferences become straight lines – not vice versa (!).

In summary: the student is geared up with two ideas of curvature: one curvature is controllable and perceivable (movement's) and the other is non-controllable and non visible (manifold or environment surface).

4.6 Everything comes from the Greek(s)?

We have found interesting to return to the etymology of the word geometry: in it is inscribed the dilemma and his solution. The original Greek sense of the word is "land measurement". It would be misleading to render it as "earth measurement", because to a contemporary reader the word earth evocates the modern concept of Earth (planet). This was not the perspective of the Ancient. Land evocates an image of flat and two-dimensional, indefinitely extended surface where figures behave as defined in Euclidean geometry. On the other side, if we stop and reflect on the overall shape of

land, as also the Greeks did, that same land is the surface of a "sphere" – what we call a planet, for which the Greeks had another conception and name, Gaia. Geometry appears to be caught in a "schizophrenic" conundrum: the land is at the same time flat and endowed with plane Euclidean geometry and curved and requiring a different geometry. As we have tried to show, this dilemma finds a deep solution in the local point of view of Riemannian manifolds: the geometry of figures is everywhere Euclidean in a sufficiently small region (locally); the geometry of the underlying surface, that is, the curvature of space, is never Euclidean[80] – and it is it that makes geometry of figures non Euclidean on large scale. Euclidean and non Euclidean are indissolubly entangled – as in the word *geometry*.

4.7 Synoptic of the epistemological model

In the following three tables we summarize the key features of our model.

We have shifted geometry to kinematics on curved Riemannian manifolds. We could not discuss Riemannian manifolds from a metric point of view, because of the limitations of our environment. Yet, we have provided a short box on it. Movements and the straight line model are the probing means to reconstruct space structure from a local point of view. In the local point of view, the underlying curvature manifests itself in tidal effects on freely falling movements.

[80] The curvature tensor is never identically null or the spherical surface is never, even infinitesimally, isometric to a plane (Berger, 2003, p. 32)

Block: Basic key elements and features of the Euclidean & non Euclidean Geometry

Key elements
1. space, explicit object of study
2. space, has curvature
3. figures, have behaviors
4. behaviors, may be Euclidean or non
5. scale, idea of magnifying

Key features
1. Locality: In any sufficiently small region
   a. behaviors are Euclidean to the approximation desired
   b. space is flat to the approximation desired
2. Behaviors depend on the place and on the scale:
   a. moving from a place to another, geometry may change from Euclidean to any type of non Euclidean
   b. whatever the geometry of a place, it becomes Euclidean going on infinitesimal scale – equivalently, on infinitesimal scale, space is flat everywhere
3. Indicators
   a. local non Euclidean behaviors are indicators of space curvature
   b. examples: change of separation of parallels; sum of internal angles of a triangle; ratio between length of a circumference and diameter
   c. local Euclidean behaviors are indicators of space flatness

For completeness: the following elements are inherent to the Riemannian model, but were not included in it.

Block: Additional key elements and features of the Euclidean & non Euclidean Geometry

Note: the metric is a key element of Riemannian manifolds, but out of range for our students (that had no knowledge of the Cartesian plane or Pythagoras theorem)

Key elements
1. space, has metric
2. curvature can be calculated from metric

Key features
1. Locality: In any sufficiently small region
   a. the metric is Pythagorean to the approximation desired
2. Indicators
   a. example: deviation from Pythagorean metric

The experience the student has of the land is our case study, and is epistemology is:

Block: Basic key elements and features of Earth's surface spherical model - Euclidean & non Euclidean

Key elements
1. the surface (the land), explicit object of study
2. the surface, has curvature
3. straight lines, triangles, circumferences - have behaviors
4. behaviors, are Euclidean or non
5. scale, idea of magnifying

Key features
1. Locality: In any sufficiently small region
   a. behaviors are Euclidean to the approximation desired
   b. space is flat to the approximation desired
2. Behaviors depend on the scale only:
   a. Moving from a place to another geometry does not change (locally)
   b. At every place geometry is non Euclidean spherical or Euclidean on scale sufficiently small
3. Indicators
   a. Non Euclidean behaviors are the same everywhere indicating constant curvature
   b. On sufficiently large scale, there are changes of: the separation of parallels, of the sum of the internal angles of a triangle and of the ratio between the length of a circumference and diameter
   c. The excess of the sum of the internal angles of any triangle indicates positive curvature
   d. The defect of the ratio indicates positive curvature

Our epistemology of geometry is kinematic and dynamics based – in line with general relativity. The straight line is the linchpin:

Block : "Geometry explored by movements"

Straight lines are single out by three types of features:
1. static
2. kinematic
3. dynamics

Curving lines
1. Lines smoothly deviating from straight lines
2. The curving is controllable or observable within the space

Curvature of space
1. The curvature of space is not controllable and not observable from within the space
2. It effectuates non Euclidean behaviors of straight lines and figures
3. It cannot be an attribute of lines or figures of space
4. It cannot change straight lines into curving lines or vice versa

Projections (maps)
1. projected straight lines may appear curving and projected curving straight lines may appear straight

Chapter 5: The laboratory design

In this chapter we answer the research question:

3. What is a laboratory's design based on the above epistemology and transposition and driven by the above requirements?

The laboratory handouts are in Attachment 1, whereas lectures plans are in Attachment 2.

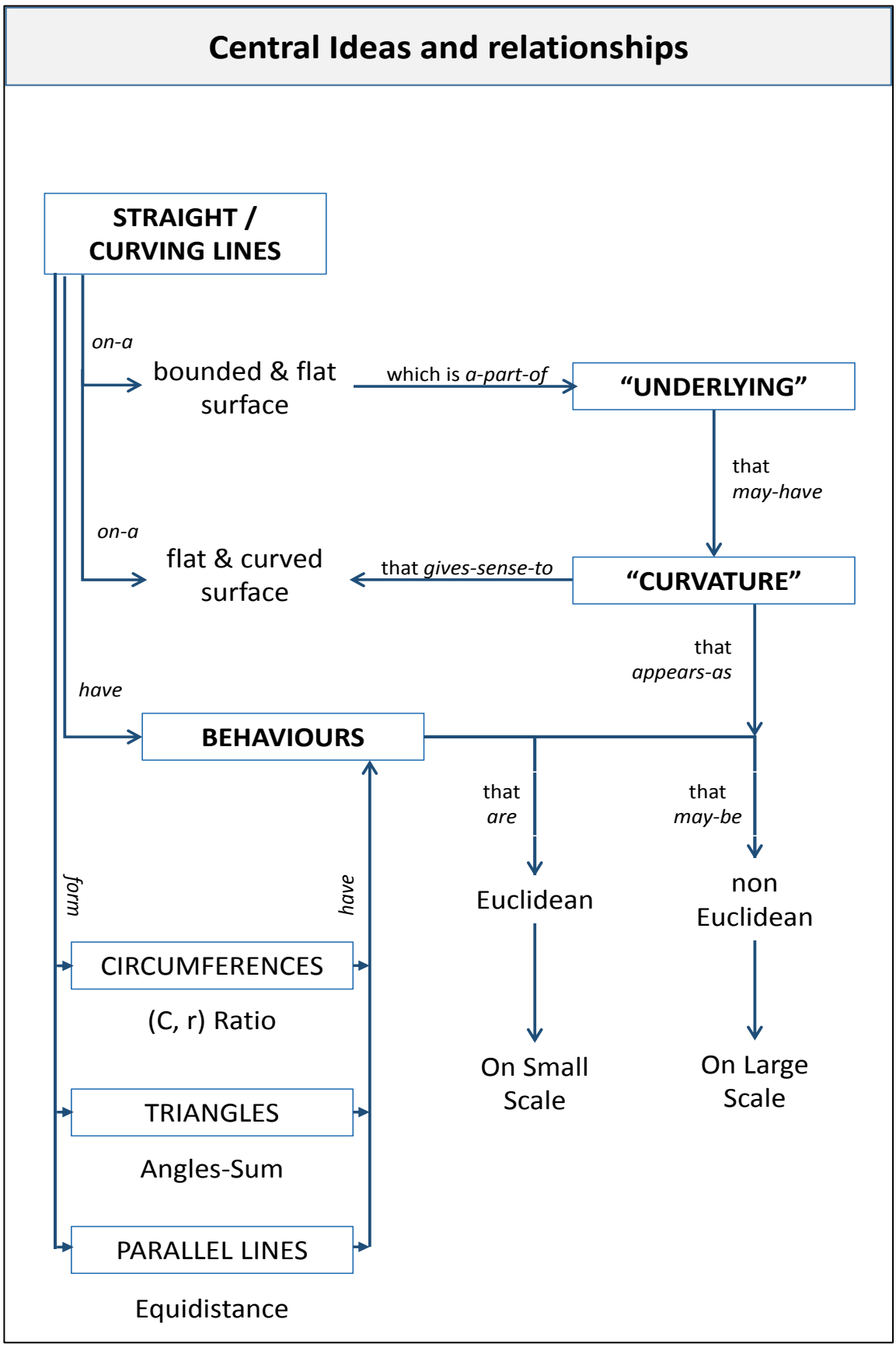
5.1 Premise

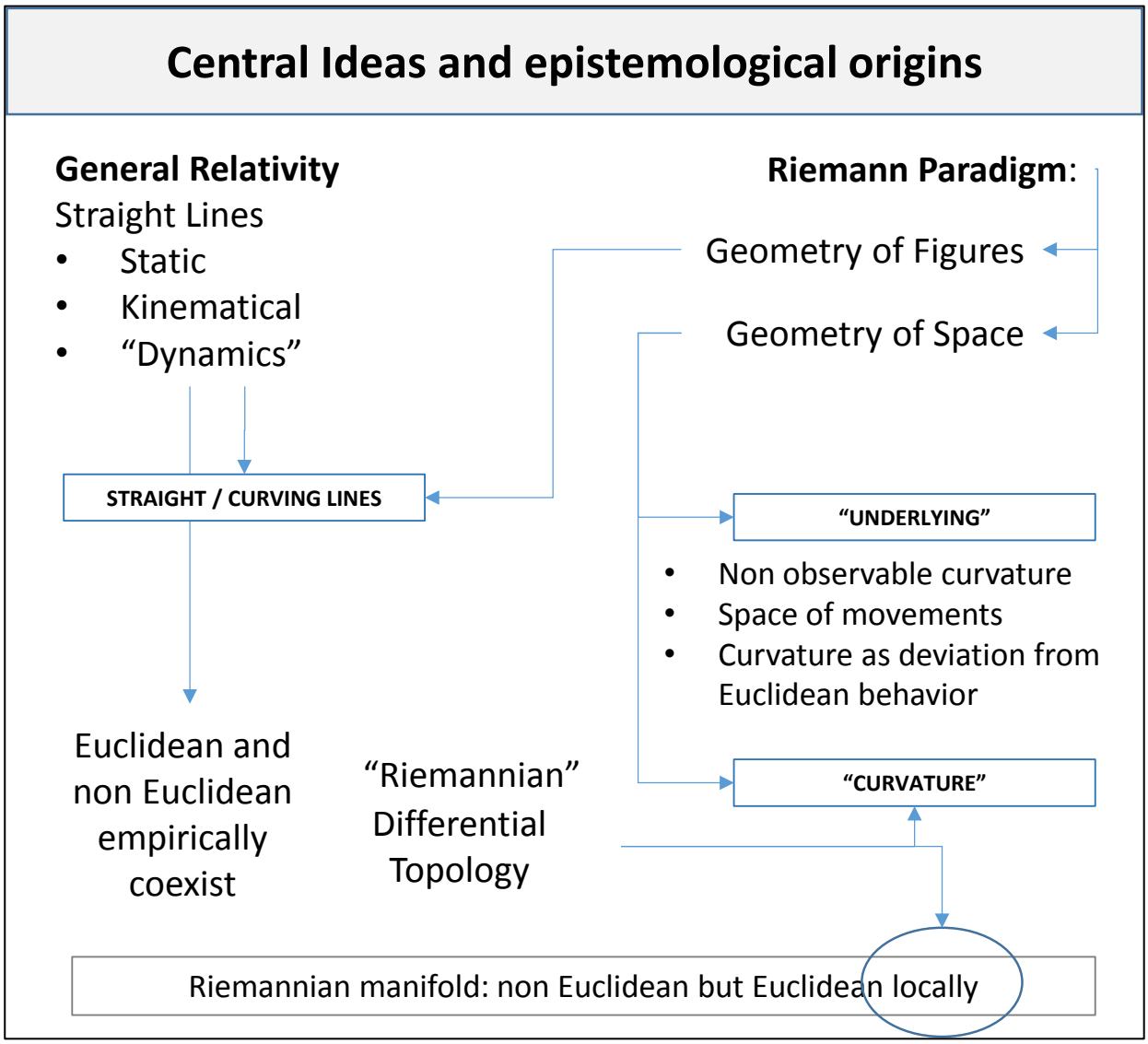
The idea is that the student works on the geometry of his land. He works on the land outdoor, by going straight and curving with bikes; feeling and observing the movements; rolling out paper-rolls; tracing lines; observing the landscape and the horizon; orienteering, etc. He works on models of the land: photographs of large flat extents of land; 2D maps of his neighborhoods and larger regions; videos of sails disappearing beyond the horizon; photographs from space of the curved land; Google Earth animations; 2D world maps; 3D world maps: spheres and the Lénárt sphere.

The key binding is between the land and the sphere: he is able to transfer observations and discoveries either way. Yet, the student work is directed by one-sided objective: learning about the geometry of his land. Learning about the geometry of the spherical surface is only a means or a secondary objective.

5.2 Central ideas and interdependencies

In our approach, students need only two ideas: "going straight" and "curvature of what is underlying the movements" – as represented in the figure of the next page, and originating in general relativity and Riemannian manifolds, as represented in the successive second schema. We are going to discuss in detail the ideas and relationships.


Central Ideas and relationships
STRAIGHT / CURVING LINES
on-a
bounded & flat surface
which is a-part-of
"UNDERLYING"
that may-have
on-a
flat & curved surface
that gives-sense-to
"CURVATURE"
that appears-as
have
BEHAVIOURS
that are
that may-be
Euclidean
non Euclidean
On Small Scale
On Large Scale
form
have
CIRCUMFERENCES
(C, r) Ratio
TRIANGLES
Angles-Sum
PARALLEL LINES
Equidistance



Straight Lines

In the previous chapter, we explained the model. Differently from common approaches, straight lines need not be redefined or justified: they are the student straight lines, those of his "Euclidean" experience – it is their behavior that will change, whereas remaining in the same space.

The only requirement is that the students expand the narrow concept of straight line of school geometry into our model of straight lines. This is done in Didactical Unit 1.

Leaving geometry as "mathematics" to embrace it as "physics of movements", and equipped with this cognitive model of straight line, the student will discover by himself amazing behaviors of his straight lines. For example: diverging straight lines will converge, cross and come back; there are

infinite couples of points for which there are infinite straight lines connecting them. This is done at the end of Didactical Unit 2.

Curvature

Students already know the meaning of "curving" and we institutionalize it in "turning the steering wheel of a bike" – the concept of a "curving line" is therefore built in contraposition to that of a straight line. It is a "curving" that is controllable and observable (differently from "space curvature" or "curvature of the underlying")

Students have to develop the meaning of "curved underlying". First of all we underline that it does not come from an act of movement – it is something which is inherent to a surface, to a place where movements happen. In Didactical Unit 2, we ask to students first to feel it, with their hands and bodies; we do not ask them to look at it "with their eyes" (because figures "do not see" the curvature). Then, of course, we ask them to discuss its appearance on different objects. Yet, when we get to the key Riemannian point of vanishing curvature, we revert to sensorial: we ask students to "feel" that the Swiss Balll feels flat to a finger ("as does the Earth for small triangle").

Circumferences

As for the straight lines, there are no different definitions of circumference for Euclidean or non Euclidean geometries: the definition is always that of his Euclidean world.

The only requirement is that the student build on a flat surface the simplest kinematical model of a circumference (two students connected by a rope and one of them going around the other).

Equipped with this kinematical model the student will discover by himself the surprising behaviors of his circumference, simply by imagining the rope longer and longer and checking it on the Lénárt sphere. The circumference becomes a straight line and there is no need any more to have the student being roped to the other! And vice versa: all straight lines have the property of being circumferences (as defined on the flat land – no need to look at them from outside the surface). And, eventually, these straight circumferences are the longest possible circumferences – for the rope shows that they get shorter, if the rope gets longer or shorter of one quarter of the distance between the center and its antipodal point. It is the student that at this point arrives at establishing that the great circles are straight lines. And not as "sections of a 3D object", or as "minimal paths", but just as the "circumferences" of his flat land. This is done in the Didactical Unit 3.

Triangles

For triangles, there is no requirement whatsoever: they are just the triangles of his flat land.

The discovery is that because straight lines never diverge, always come back to cross, one can "close" impossible "to be closed" triangles.

What enables all the key insights is the ability to "think" on two levels: geometry of figures and geometry of space (the underlying) - it is the curvature of the "underlying" that "closes" impossible "to be closed" triangles. The student is already trained on it, for, already in Didactical Unit 1, he was trained to split bending of lines and bending of the underlying – by working with the double concept of curvature.

The above aspects are a radical departure from common approaches. For example, many start from great circles, typically, by asserting: "a great circle on the sphere is a straight line on the sphere" – and providing some justifications like: they are the simplest lines; they are the line of cutting in half an orange; they are the path of stretching elastic bands; they are what an ant would see (for example, Lénárt, 2013). In line with our psycho-educational requirements, our students are not concerned with straight lines of a sphere, but with straight lines of their world. Therefore, they do not need them to be defined – they know what they are, and all the rest comes consequently.

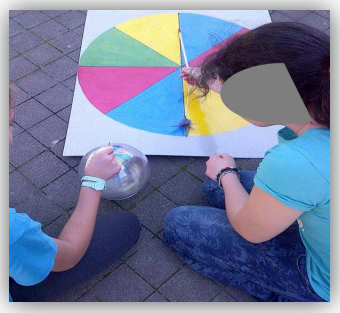
5.3 The course

In the Didactical Unit 1, students start out by building the concept of straight line as a straight line (static), a straight movement (kinematic) and a stress (force)-free movement (dynamic). The concept is constructed initially on a surface (2D space) empirically flat and bounded. The surface is the basketball court and the students work in first person by creating the movements. They use bikes, arrows and other tools.

In the Didactical Unit 2, the student develops awareness that the "horizon bounded region of the surface in which he lives" is flat and in it parallel lines keep being parallel, triangles and circumferences are Euclidean – no changes with respect to an A4 paper sheet. However far he goes he will keep observing the same facts – on the same scale. He works cognitively on the fact that there is no point after which the surface curves. He uses videos, pictures and software. Then, the student works on perceptive cognition of curvature to build an idea of "something" which "drives how our hands and body moves on a surface" (with Swiss Balls and other objects). Curvature is

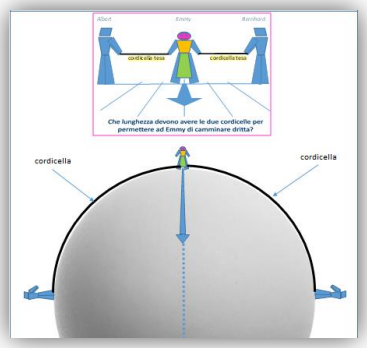
discussed for a spherical surface and the key dilemma – “how is it that the land is flat and yet is curved”, is discussed. The student is led to develop a paradoxical thinking. To prepare the stage for the last part of Didactical Unit 2 (the discovery that the “Euclidean” geometry of figures is changed into non Euclidean by the curvature) the student is asked to spell clearly his convictions about (flat) geometry of straight lines (a questionnaire).

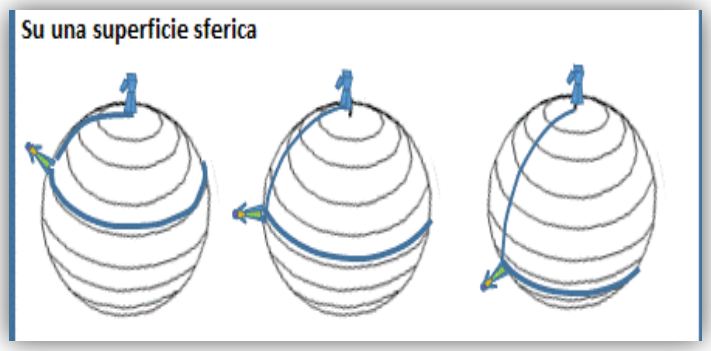
At this point the key binding between the “world” and the Lénárt sphere is made. The flat bounded region is “mapped” onto a spherical model of the Earth (globes and Lénárt spheres) and the student and its “straight lines” are avatarised on it (we used two tools – called the “papalina” and “papalona” – see the plan of the Didactical Unit 2 – two circular patches with the same colors – one on the sphere and the other on the ground).



The concept of straight line, straight movement and stress-free movement are ported on the spherical surface and the student works out how the straight lines of his land should look like on the spherical surface – if he applies the straight line cognitive model.

Starting from the end of Unit 2, the student discovers non Euclidean facts on the spherical surface that are translated in consequences for his world: going around in circles around a point by longer and longer radii leads to going straight and then to reduce the circular paths to a point (Unit 3);





triangles impossible to be closed, because the sum of the internal angles exceed one flat angle, do exist (Unit 4); extending parallel lines will make them cross (Unit 5).

The surface is both flat and non flat. Straight lines, triangles and circumferences behave both in non Euclidean and Euclidean ways – depending on the scale.

Discoveries are proposed also by "socio-empathic" situations: the student discovers real things about displacements and separations from his friends on the land (e.g., he has to discover that whatever direction he goes off, he will always arrive at his friend's place – if they are antipodal).

Cognitive conflicts are continuously used to highlight contradiction between Euclidean and non Euclidean behaviors. Cognitive conflicts are resolved by "fusion" (Euclidean and non Euclidean), not by "relativization" as in traditional non Euclidean geometry (that lacks of the Riemannian concept of "locally Euclidean"). Cognitive conflicts are wrapped into socio-affective empathic quiz's – in which the student may empathize with the character and "feel" the problem and root the solution in feelings.

Every geometrical concept is worked out on a multiple sensorial perspectives – visual and kinesthetic, integrated in logico-linguistic.

As we discussed in chapter 2, the visual intelligence is critical, because the whole work is constantly requiring processing of images and drawings. The kinesthetic intelligence is critical, because the whole work concerns the student perception of himself & of the space (land) to which he belongs (proprioperception and orienteering). Dynamics of the interplay of these intelligences follows dynamical system theory.

Yet, the conceptualization of concepts as complex as curvature, requires integration by logico-linguistic intelligence – leading to structured knowledge of the form that makes up the current transmissible form in which mathematics is set[81]. Work groups are always combined to ensure coverage of this important part of intelligence. Normative activities are reduced to a minimum and space for students to develop their own strategies is left.

[81] One could imagine that in a near future current mathematics texts be superseded by augmented reality holistic (total cognition – kinesthetic, visual, etc.) environments.

For the students that are already in the stage of formal operation, by meeting, dealing and eventually making sense of logico-linguistic contradictions, as “flat and curved at the same time”, rooted in semantics (“meaning in the world”) - not in merely syntactic playing (as those based on definitions), they may develop a cognitive level that help them overcome limitations of our language, models and logic.

The student is motivated to push forward his logico-linguistic intelligence also by the need to communicate his feelings and perceptions to his peers.

Multiple work streams

The “cognitive and geometrical contents” are articulated by five different streams: experiments, static geometry, kinematics, emphatic situations and applications – consistently with our psycho-educational analysis and epistemology.

The figure in the following page contains the schema we used in the design phase – with the sample key elements.

Earth’s model

An essential part of our work has been about bridging the gap between the physical experience students have of the land and the model of it (a simple spherical surface). Further to having to cope with a large variance of preexisting knowledge about the shape of the Earth, we had to lead students to recognize that there are as many models as problems and that ours responds only to a “purely geometric” problem. The modelling issue is critical to our laboratory, because students have to “feel” (identify) the model as really representing their world. Unit 2 deals with the Earth’s model.

THEMES / WORKSTREAMS	Retta	Curvatura (del manifold)	Circonferenza e rette	Circonferenza e metrica	Triangoli	Parallele
Esperienze/ Sperimentazioni	*GE: Bici, Freccia, Rotolo, Nastro, Altro, Cerchione* *GNE: Cerchietto, Sfera, Globo, Stuzzicadenti, Biadesivo; Cilindro e Sella (*)*	*Sdraiati su pavimento, poi di pancia su swiss ball, dito su swiss ball, calotta sferica, su cilindro e su sella*	*GE: Corsa circolare con corda, allievi con raggi, rotazione a braccia sospese, gesso e corda* *GNE: Mirroring su sfera distanza costante da punto fisso*	*Anelli di ferro.*	*GE: triangoli sul sedime con bicicletta, nastro adesivo, ecc. sempre più grandi* *GNE: Mirroring sulla calotta. Triangoli col nastro sulla swiss ball. Calottina e disco.*	*GE: due bici con frecce,* *GNE: Cerchietto, Sfera, Globo, Stuzzicadenti, Biadesivo*
Sapere Geometria Cinematica	*GE: un solo cammino da P a Q per qualsiasi P e Q* *GNE: due cammini e: esiste un Q per il quale ci sono infiniti cammini da P*	*I comportamenti cinematici NE dei personaggi sono interpretati come effetti della curvatura del palco su cui recitano: un «sottostante» - non accessibile (manifold)*	*GE: per girare attorno ad un punto devo per forza curvare* *GNE: ci sono due punti attorno ai quali giro andando dritto*	*GE: per girare intorno ad un punto, ci vuole tanto più tempo quanto più sono distante dal punto* *GNE: oltre una certa distanza il tempo diminuisce!*	*Da definire se far rotazioni in GE e GNE*	*GE: due moti che partono paralleli e rettilinei non s'incrociano mai* *GNE: due moti che partono paralleli e rettilinei s'incrociano*
Sapere Geometria Statica	*GE: una sola retta per P e Q qualsiasi* *GNE: una sola o infinite rette per due punti P e Q*	*Gli aspetti statici dei comportamenti sono egualmente interpretati come effetti della curvatura del sottostante non accessibile*	*GE: le circonferenze sono sempre linee curve* *GNE: alcune circonferenze sono linee rette*	*GE: quanto aumenta la circonferenza dipende solo da quanto aumenta il diametro* *GNE: la circonferenza prima aumenta e poi diminuisce all'aumentare del diametro*	*GE: la somma degli angoli interni di un triangolo è 180° e non dipende dalla sua grandezza o forma o altro* *GNE: la somma degli angoli interni aumenta con l'area ed è > 180° e < 540°*	*GE: segmenti «equidistanti» si prolungano in rette equidistanti all'infinito dette parallele* *GNE: segmenti «equidistanti» si prolungano in rette che s'incontrano*
Enigmi Empatici	*Albert va dritto, non sa dov'è Emmy, ma qualsiasi direzione prenda arriva sempre da Emmy. Dove si trova Emmy?*	*Gli enigmi sono spiegati in termini di curvatura del sottostante non accessibile*	*Emmy è legata a Albert e Bernhard (che stanno fermi) da due corde della stessa lunghezza. Quanto sono lunghe le corde se Emmy può camminare dritta senza fine?*	*N e F giocano a chi fa circonferenze più enormi. F dice «sempre il tuo diametro piu' 1m! Ho vinto!» N, sceglie un certo diametro e poi dice, «la tua circonferenza è più corta!»*	*Emmy ha litigato con Albert: va via dritta fino ad un grande albero; gira a destra (90°) e va dritta fino ad pozzo; gira a destra (90°) e va dritta finche' incontra Albert che non si era mosso. Quanti chilometri dista l'albero?*	*Emmy e Janos hanno due aerei e volano uno di fianco all'altro. Emmy dice che lei va dritta. Janos che lui curva. Com'è possibile.*
Paradossi Cartografici, ecc.	*GE: andare dritto corrisponde alle rette sulla cartina e v.v.* *GNE: andare dritto non corrisponde a rette sulla cartina (in generale)*	*I paradossi sono spiegati in termini di curvatura del sottostante non accessibile*	*GE: rette sulla cartina sono rette sul terreno* *GNE: i meridiani sono linee rette, i paralleli sono curve (equatore escluso)*	*TBD*	*GE: i confini di una regione hanno la forma di un triangolo equilatero* *GNE: come sopra ma gli angoli sono di 90*	*GE: rette parallele sulla cartina corrispondono a cammini paralleli* *GNE: rette parallele sulla cartina non corrispondono a cammini paralleli*
Altro	Cilidno, sella, superficie a curvatura variabilie e mini-toro (Briko)					
Storia e Uomini	Euclide, F. Gauss, J. Bolyai, N.J. Lobachevsky, B. Riemann, A. Einstein, A. E. Noether					

5.4 A look at other works

Our work "is based on psycho-educational requirements and the student works outdoor and using artifacts discovers that his world geometry is non Euclidean Riemannian, i.e.: locally Euclidean and globally non Euclidean.

We have found no similar laboratory. Other laboratories study the geometry of the sphere and from that may infer facts about the geometry of our Earth's surface. We have found no laboratory having students build a complex straight line as ours and being driven by psycho-educational requirements instead of by the body of knowledge/ geometry. Our student ego and sensorimotorial centric work involving perceptions, schemes, images, personal models, artefacts and geometric conceptions – and working on the sphere as an avatar, appeared very particular.

We have found no similar epistemology in other works: that is, physics of movements – that by respecting general relativity equivalence principle generates non Euclidean geometry from locally Euclidean geometry. Consequently, we have not found any embodiment theory that "embodies" such a vision of geometry.

The mainstream follows a "comparative geometry" approach – on one side Euclidean geometry, on the other non Euclidean and compare one with the other. Geometry is part of mathematics – not of physics. Although we recognize the merits of such an approach, we believe that by counterbalancing it by an epistemology and approach similar to ours would also be beneficial to clarify the general debate about whether the world is Euclidean or non Euclidean.

We think that the "diversity" of our work is a simple consequence of our having questioned the body of knowledge of traditional non Euclidean geometries to find a way to fulfill our students needs – though we have met significant difficulties in transposing Riemannian manifolds and general relativity concepts in our epistemological analysis.

We do not discuss how our laboratory deals differently with the details of curvature, parallel lines, circumference, space, etc. – because we have already partially done it in the epistemological analysis and design of the laboratory.

That said, there are many texts and laboratories on non Euclidean geometry from which we have learnt a lot. Secondly, there are two major streams of experimentation with non Euclidean geometries that provided us with inspiration, a very rich baseline and a benchmark.

The first is the work of Lénárt - essential because it provided our students with artifacts optimized to carry out geometry on a spherical surface. The Lénárt course on non Euclidean geometry covers

a much wider set of spherical geometry facts and applications than ours. Naturally, Lénárt work's vision of non Euclidean geometry and objectives for students are different from ours, for he follows common epistemology of the Many Geometries and students discover geometry of the sphere and, secondarily, apply to the Earth. But, Lenart's kit also provided us with a very effective laboratory for the teacher in which to explore "epistemology".

The second work is that of Arzarello and associates (Arzarello et al., 2012) The concept of straight line there presented is very similar to ours – not surprisingly, because they take a vast view of geometry encompassing also differential geometry and general relativity. Their work is intended for high school and, consequently, neither develops our psycho-educational requirements, nor develops the concept of straight line into a model to guide ego/bodycentrical cognition (embodiment) building on the land. Yet, it is clear from their intents (p. 2), that their pedagogical and didactical philosophy is also in the direction of socio-constructivist learning situations with a central concern for meaningfulness (for the student). Their work covers a vast range of topics and applications. Yet, it follows the paradigm of the Many Geometries – even in a wider perspective: as explained in the preface and introduction, they consider a multiplicity of geometries of mathematics, further factored in projective, analytical, etc. This is a perspective centered on the body of knowledge (the many bodies of knowledge) - in line with the work intent of being a reference for the design of a variety of laboratories within the overall mathematical sciences context.

Chapter 6: Observation of students' response

In this chapter we focus on our fourth research question:

4. What is the response of the students, in broad terms of cognition and meaning, to the implementation of such a laboratory?

6.1 Targets

We defined as primary targets of our observation the:

1. Concept of straight line
2. Concept of curvature
3. Non Euclidean behavior of circumference
4. Non Euclidean behavior of triangles
5. Non Euclidean behavior of parallels

For the observation, these are the cardinal elements of the geometry based on our epistemology.

Secondary targets concerned:

A. how students responded to a kinematical work of geometry – in contrast to the more traditional static
B. to which extent the students would relate their learning to their experience of the world
C. what opinion changes were expressed by the students about geometry and mathematics

We observed these aspects because of their relevance within our psycho-educational requirements.

6.2 A priori limitations of the written tests

Discussion

As discussed in the psycho-educational requirements and in the laboratory design, a profound and effective learning of geometry requires working at the sensorimotorial level of cognition. Although we have put this level as central concern of our work, in our opinion, the logico-linguistic (and visual) cognition, intelligence or competence, becomes the key and weakest "ring" in the "process".

Let's distinguish, for simplicity, the knowledge building process from the observation process. For the former, the logico-linguistic intelligence is the key ring for structuring and integrating that

cognition (knowledge building) derived from the sensorimotorial experiences. As noted also in several points and in our detailed observations (see Attachment 2), it is indispensable to structure sensorimotorial and visual cognitions to subsequently process them by abstraction and basic hypothetical-deduction mechanisms to finally arrive at structures of mathematical geometry. Indeed, in the laboratory we cared about having students to constantly verbally re-express kinesthetic and visual work (cognitions) and about having them wrapping up the concepts in the written verbal register - often using paradoxical logic (see also the handouts in Attachment 1).

But, it is in the observation process (the participative observation carried out by the teacher), that the logico-linguistic competence shows its critical weakness. This competence is the principal means by which the student externalizes his knowledge and cognition and by which the student cognitively processes the teacher requests or statements. The average student logico-linguistic competence, is sufficient to communicate in customary learning situations of Euclidean geometry, but breaks down in our learning situations in which the student has to discuss kinesthetic, orienteering and new concepts such as curvature as well as restructuring previous geometrical concepts in more general forms and paradoxical logic by the use of language. Moreover, at the evolutionary stage of our students the logical abilities are still forming and their verbal expression capability is still limited. All these elements also amplify the differences and range of variance of the students' profiles (see also next paragraph 5.6).

We have already discussed how to deal with this weakness in knowledge building in the chapter about the laboratory – which is also discussed at detail level in some sample situations reported in the observations included in Attachment 2.

For all these reasons, we have decided to avoid traditional written tests in favor of an observation in "nonverbal information augmented" situations: collective or individual "discussions" in which the student is encouraged to use mimicking, objects, movements, inventions of stories, etc. to respond to teacher questions. In "holistic communication" we read his "verbal expressions"[82]. In other terms, adopting a semiotic perspective, we believe that the students expressions in the verbal and figural registers permitted in a written tests, cannot be understood (given a meaning) without

[82] Our position is consistent with the conception of mathematics adopted for this work. If one was to take a more logico-formalistic conception, we suppose, « the learning ».would be largely or entirely reduced to the ability to logico-linguistically express the concepts.

considering the class, with his history of lectures, laboratories, group dynamics, cognitive profiles, didactic contracts, etc. Indeed, as argued by Radford:

"[…] I processi di significazione […] non sono semplicemente realizzati attraverso il simbolismo matematico. […] intervengono […] i gesti, le parole, la intonazione, il ritmo e altri segni corporali. […] la matematica appare come riflessione e azione specifica sul mondo, realizzata in e attraverso i segni mondani, corporali e scientifici […] creando così reti complesse di significati" (Radford cited in D'Amore et al., 2013, p. IX)

Three observation channels

Therefore, the observation process consisted of two primary channels and one secondary:

1. interactive observation - observation of and interaction with students during the laboratory: a primary, non structured and continuous process

2. bilateral interviews – a relaxed version of traditional "clinical interviews": a primary, semi-structured and discontinuous process

3. written tests – in the classroom and generally individual: secondary, structured and scheduled (only two tests)

We discuss each channel and report the observation results, starting with the interactive observation.

6.3 The interactive observation

Rationale

It was our primary strategy for the reasons explained in the previous paragraph and because:

Socio-affective – in contrast with written texts, during the work in the laboratory students would feel more and more at ease and not being judged or evaluated; this allowed us a 360° view of their attitudes, cognition, knowledge and competency in a real situation (in action) - problematic and socio-constructivist

Integrated – during the work in the laboratory the student's reasoning, difficulties, successes, etc. are understood in the dynamic network of their relationships with what happened and was said

before, what was said by a colleague, what the student was looking at, his expressions, his attitudes, etc. rather than having “isolated data” – fragments.

Target focus

Recalling our target classifications (paragraph 5.1), the interactive observation targets were both primary and secondary.

Data capturing/ processing

For data capturing we considered/ used:

Video Recording – the very first hypothesis was to video everything, for we had thought that audio recording would be missing all the nonverbal facts. After assessing the logistic complexity of finding and operating at least four cameras (we had 22 students), we abandoned the idea in favor of audio recording.

Audio Recording – our initial aim was to record all. Yet, after the first lecture we realized that it was ineffective, because even with two recorders the background noise would make the recordings very difficult to reprocess and we would not have any notes and little “memories” (having entrusted “mass memorization” to the devices)

Taking notes – we had to do it synch and asynch (completing notes within one hour from the end of the lecture); the in-service teacher was able to assist some lectures and to take notes to help us with.

For data processing we endeavored to process all of our notes key points before the successive lecture, so to: interpret them, make hypothesis and test them in the successive lecture (for the most important points).

For the primary targets, we used tables to score interventions – both on paper and e-worksheets.

For the secondary targets, we simply kept our notes.

Primary and secondary observations

Traces and non structured observations captured in each lecture are reported in Attachment 2. For the rest, we proceeded as follows.

For the primary targets (shortly, for primary observation) we structured a framework.

Primary observation framework

First we defined the concept[83]; then we defined the indicator and finally the observable behaviors.

Let's start with the "concept of straight line":

Concept of straight line: "The students "isolate" the features intrinsic to the movement from the features of its trace".

Indicator: "The student distinguishes a straight movement as the one in which the steering wheel of the bike is kept straight – with no concern about the appearance of the trace left; in particular: he does not let himself being misled by a trace on a curving surface (e.g., spherical) as an indication of a "curving line"".

Observables (that are already levels of learning)

L1. student's basic verbalization of the concept with nonverbal scaffolding – Helping himself with gestures and artifacts and citing examples, the student can make himself understood by the teacher and appears (to the teacher – subjective) to have captured the essence of the concept

L2. student's appropriate verbalization and representation of the concept – As above, but with reduced scaffolding and, in addition, the student can propose more examples or a more structured example and/ or use an artifact in a more structured and richer way

L3. student's own examples or ideas expressing the concept – As above and, in addition, the student appears to have added personal interpretations, models, images, situations, etc. to those seen in class

L4. student's counterexamples or use in other concepts – As above and, in addition, the students can use the concept to answer a "problematic" situation

We have formulated the observables to be independent from the specific concept, so that we will reuse them for the other four target elements - for which concepts and indicators are:

Concept of curvature: The student "isolates" the features proper of the figure from the features "proper" of the space. Indicator: The student distinguishes the curving of the line from the curvature of the surface.

[83] These are operational definitions to focus the observation – not a substitute of the epistemological discussion.

Non Euclidean behavior of circumference: The student “isolates” the feature of being at constant distance from a given point from the appearance of a curving line. Indicator: The student recognizes that going straight on the land is describing a circumference.

Non Euclidean behavior of triangles: The student recognizes the Euclidean behavior as characteristic of the flat land. Indicator: The student knows examples of non Euclidean behaviors.

Non Euclidean behavior of parallels: The student recognizes the Euclidean behavior as characteristic of the flat land. Indicator: The student knows that no straight parallel segments can remain so on large scale.

Primary observation process and results

We assessed the observables of the five key concepts - across the four learnings streams: static geometry, kinematics, emphatic situations and applications, and during the work in class. The assessment was always based on the teacher subjective interpretation of the individual students and student groups work and discussions. The teacher would “tag” the topic and the student(s) name in his notebook – during or after the lecture. Specific collective sessions (maximum 5 minutes long) would be taken at the end of each lecture for the teacher to “tag” the learning level. All the tags were consolidated in a simple matrix showing for each student and topic the estimated level of learning. The resulting statistical distribution is in the following graphic accompanied by a table with the number of students by level and concept.

Graphic 1 – Number of student by levels and concept

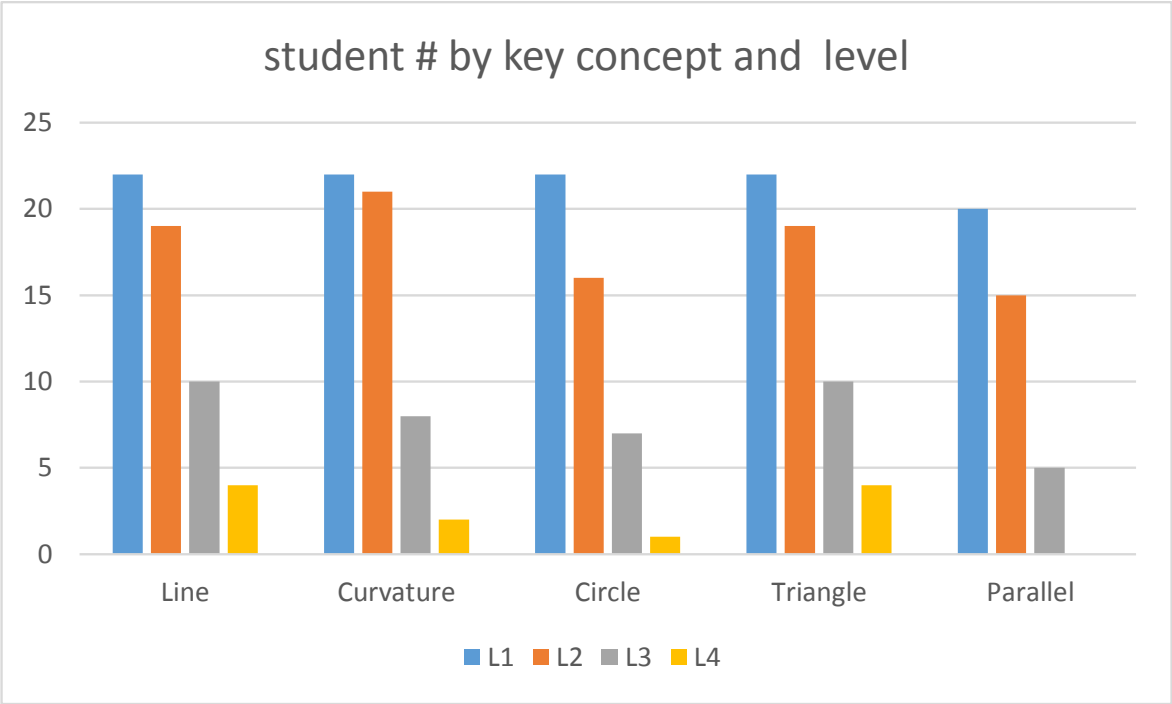


Table 2 – Number of student by levels and concept

	Line	Curvature	Circle	Triangle	Parallel
L1	22	22	22	22	20
L2	19	21	16	19	15
L3	10	8	7	10	5
L4	4	2	1	4	0

Traces and non structured observations captured in each lecture are reported in Attachment 2.

Results discussion

a. All students reached L1 – except 2 students for the topic of the “parallels”. We have interpreted it as due to the epistemological difficulty of the topic (also discussed in the Attachment 4).

b. Passing to the L2, a very high number of students have reached L2 for the curvature and for the straight lines. Our interpretation is that by having presented both topics in a highly sensorimotorial or kinesthetic style, we have called in the most primordial and mature cognition layers of our students. In addition, with this approach the laboratory was much more “ludic” for the students, compared to a “sitting at the desk approach”, and facilitated learning by creating the appropriate socio-affective conditions. Finally, the transposition did not present to us the challenges of the parallels.

c. The number of students at L2 for the “triangle” is the same as for the “line”. We interpret it as an indication that the topic is easier – having spent on it just one lecture.

d. For the circumference, the number of L2’s is lower. Our interpretation is that the topic was very “counterintuitive” (we have found it ourselves the most counterintuitive) and that the time was too short for the students. The circumference has, indeed, appeared a difficult topic also in the exit test.

e. For the L3, we can observe that it was reached by 10 our 22 students for both the line and the triangle. Our interpretation is that for the line it was reached because we carefully designed the laboratory on it and we allowed for sufficient time - whereas for the triangle it was reached because it requires a cognitive process more similar to those of traditional geometry – in which, indeed, more than half of the class is generally above the average.

f. For the curvature and the circle L3 is slightly less. We may interpret it as an indication that traditional cognitive processes are not enough: curvature may be easy to grasp in a superficial way, but its difficulty grows exponentially with the deepening of the concept (as it is discussed in Attachment 4). For the circumference, it applies what already noted above.

g. For the parallels, L3’s plummet at 5. We interpret the data as effect of the essential epistemological and didactical difficulty – already mentioned.

h. Finally, L4 was reached both for the Line and the Triangle by 4 students. We interpret it following the same arguments made at point e. This number decreases passing from the curvature, through the circumference to the parallels. Our interpretation is that, though it is possible that this be an effect of a comparatively higher difficulty, we are more inclined to think that this could be an effect of our difficulty in assessing the students on such a complex “use case” (learning situation): we would have needed a more carefully designed clinical observations.

Secondary observation results and discussion

We recorded observation on spot. For the following second (B) and third (C) points, we had planned to collect data also by a final questionnaire, but there was no time.

A. “How students responded to a kinematical work of geometry – in contrast to the more traditional static”

We observed that:

1. The students had an initial difficulty to consider the kinematical point of view. Our subjective interpretation is that there were two main reasons: students believed that mathematics is about figures (static); the didactic contract would hinder changing that mindset; the students and the teacher did not have a common vocabulary to talk about kinematics and that made the communication difficult
2. Once the didactic contract and the communication obstacle were overcome, more than half of the class would use kinematical thinking to express their understanding and solutions. The other students appeared emotionally more comfortable with "hanging on" traditional static images. We did not have the tools to ascertain to which degree such an attitude was indicating a psychological need to hang on more structured and validated images or an effective rigidity or even a different cognitive root attitude. Yet, in the end, almost all students could process situations kinematically
3. In particular, eventually, almost all students would use kinematics in all situations concerning the identification of straight lines on the sphere or on our terrestrial surface and its 2D maps
4. All students used a kinematic explanation to successfully explaining why a straight line on a curved surface is a straight line

B. "to which extent the students would relate their learning to their experience of the world"

We observed that

1. as for the concept and properties of straight lines, almost all students would situate them on their land (especially through the empathic situations)
2. The same is true for the curvature
3. For the non Euclidean behavior of triangles, we observed that more than half of the students would speak of triangles on their land. For the rest, we do not have evidence in either sense
4. For the circumferences, our observations do not provide clear evidence. Indeed, while the largest part of the students appeared to situate the straight circumference on the land, by vivid and concrete images based on the empathic situation we led them into, very few students were able to express in a clear fashion that a circumference drawn on the ground will eventually shrink to a point
5. For the parallels, most of the students expressed clear images and stories about the fact that every two initially parallel lines of their land would eventually cross; yet, very few were

able to discuss parallel lines to a finer level directly in terms of their own examples on the land

6. Most of the students followed a three stage process: initially refused and express incredulity about the paradoxical effects of non Euclidean behaviors; then accepted them; and finally played with them, seeking other paradoxical effects (sometimes not correct) on their land and world of relationships

D. "what opinion changes were expressed by the students about geometry and mathematics"

All students expressed their surprise about doing geometry in the relative more holistic perspective we proposed. For many of them, the traditional image of geometry was so strong that there was no change of opinion: our laboratory was simply not mathematics. Eventually, most of them accepted it as mathematics, characterized it by different expressions, including: "discovery", "you understand what is outdoor", "mathematics is about the planet and astronomy", "geometry makes us see what we cannot see", "it is great fun", "we do not get bored" - and all expressed positive emotional feedback about the type of work.

6.4 The bilateral interviews

This means of observation was used when a key point – an obstacle, an idea, an issue, etc., emerged in one or several students. We relaxed the requisites of "neutrality" (or "non interfering observation"), for it was not feasible for one teacher only to switch from "accompanying" a student in difficulty to "clinically observing it".

Data capturing was done by notes, generally taken by the in service teacher and completed together after the interview.

Data processing followed the same timing and objectives as for the interactive observations (immediately after and used to fine tune next lectures).

Reports of some extracts of these interviews are included in the one-page of observations for each lecture in Attachment 2.

They were not processed per se, but were consolidated directly into the interactive observation results.

6.5 The written tests

A priori limitations

As discussed in the paragraph 5.2 and in our psycho-educational requirements, geometry cognition is sensorimotorial, visual, body-centric, observer participating and logico-linguistic. In a written test, this whole fluid cognition is, flattened and collapsed into a few elements expressed in the verbal (linguistic) written register and in the figural register (D'Amore et al. (2013)) – that cannot possibly render a "faithful" or "effective" image[84].

The first flattening and collapsing is done by the teacher. In our case, he takes of his model[85] what he thinks to be the target for the test and he writes and draws the questions - based on his assumptions on students' competence with the verbal and visual register; on his assumptions on students preexisting images and models, etc.

Then the student has to reverse the process. He does not see the teacher model; he sees the test paper sheets: many sets of signs (semiotics) handed over to him in a specific socio-affective situation - within a more general school situation (didactic contract), and largely varying depending on the student profile. He thinks he has to guess what is wanted from him. By his logico-linguistic intelligence and his visual intelligence (inevitably different from what the teacher assumed - the student has to back-process the questions (signs) to activate his model of straight line. In that model, he has to "navigate" (e.g., sensorimotorial perceptions, proto verbal reasoning, emotional-affective feelings) to find "the answers". Then, it is his turn, to flatten and collapse his understanding of his model into the verbal and figural registers. Because of his relative limited expressive capabilities in the verbal register – especially, written, as well as in the figural, the "loss" from the "answers" in his model and their "coding" on the paper is very significant and largely varying depending on the student's profile.

In summary, we believe that too much information is lost and that the risk that the student's answer is not relevant is high:

[84] Compare it with an interactive observation within the laboratory execution.

[85] Based on our epistemology and psycho-educational requirements.

1. loss of content - the student answers are only what he could express by his limited verbal and figural competencies of that part of his model of straight line that appeared to him the one in which the teacher was interested
2. loss of relevance – the student understanding of the question is dependent on how he decodes the verbal and figural registers of the teacher – registers that are in turn dependent on the choices made by the teacher in encoding his models into the verbal and figural register

Mitigation strategy

To work around these a priori limitations, we have decided

1. for the entry test - to target only the aspects of the straight line concept easier to be expressed in a language comprehensible to our students[86], and introduce circumference and triangles only in the exit test, after students have had some "training" – with the new approach to geometry

2. to have short interactions during the test, clarifying students doubts about the "decoding" of the questions

3. to have a post execution session letting the teacher and the students discuss and clarify their mutual understanding of questions and answers

For the exit test the teacher would use additional interpretation or decoding keys provided by:

a. a general knowledge of the jargon and students expressions and their meaning

b. a class or student specific track record of questions, interventions, difficulties and successes, etc. that help contextualize students answers

Objective

Because of the previous reasons, although we had planned the tests to be of primary role, we changed by promoting to primary role the interactive observation and by reducing to two the written tests and their scope to the concept of straight line and, partially, parallel lines.

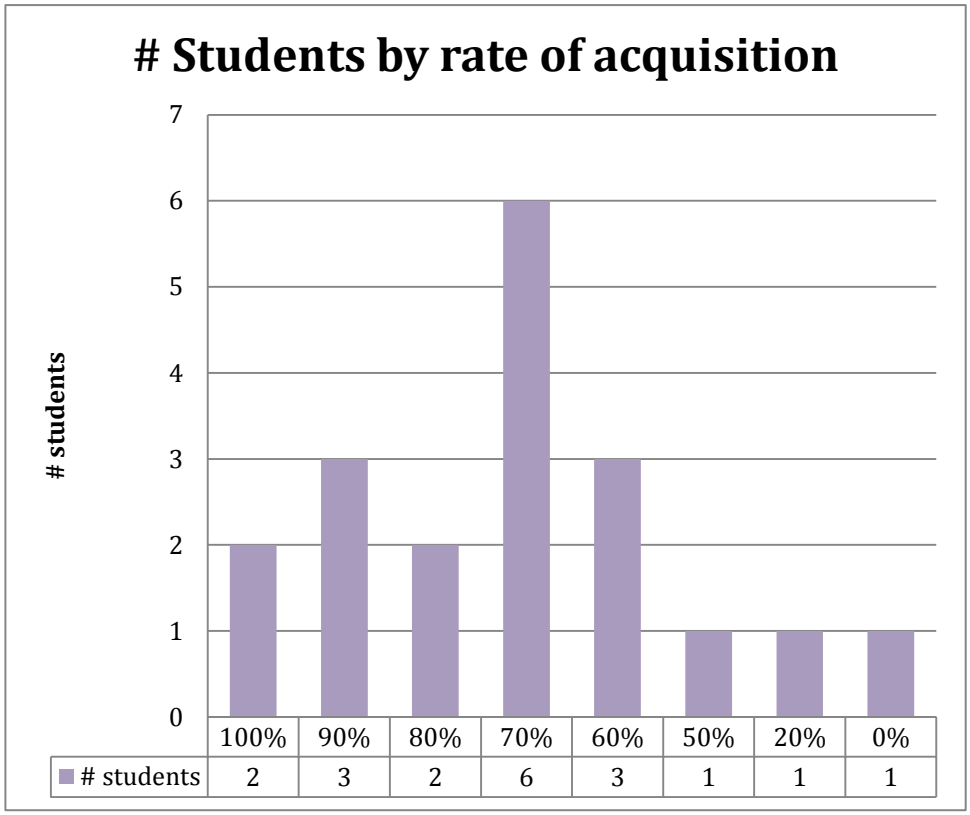
The broad objective of the two tests was

[86] All the rest is left to the interactive observation and bilateral interviews.

1. providing indications about how students changed their conceptions and convictions about straight lines – of school geometry, of the land where they live (e.g., of the basketball field) and where they are told they live (i.e., the Earth surface on 2D maps) as well as of the space in which they "imagine and live" their socio-empathic relationships
2. providing indications about how students acquired geometric cognition of the non Euclidean aspects of the circumference and of the triangles of their land and of the sphere

In addition to the exit and entry tests, we kept the diagnostic test on the Earth's model – required to assess the prerequisite to the Lecture 2, on Earth's Riemannian spherical model. Yet, the test was implemented as a collective discussion (see Attachment 1, handouts, lecture 2)

Written test results



The tests and samples are in Attachment 3. The table indicates the number of students by acquisition rate. The acquisition rate is a qualitative estimate of what percentage of the basic concepts was reached by the student. The estimation is based on the analysis and interpretation of the student's answers to the exit test – done collectively and, in some cases, bilaterally with the student. The exit test consisted of two sections. In the first section the student had to answer questions about if/ how he would change his answers to the entry test – to reflect his new learning

about straight lines and parallels. In the second part the student had to answer questions about the circumference and the triangles – pointing to non Euclidean behaviors. The rate of acquisition of new learnings was considered 100% when the student displayed, during the collective discussion following the test (in which the teacher explains his questions and the student explain their answers), a sound understanding of all topics (roughly corresponding to a L3 across all topics). A rate of 70% corresponds to a student that has shown a good understanding in at least two conceptual areas and a sufficient level in the other (roughly corresponding to a L2 and L1). The teacher estimation (assessment) process was highly subjective and based on a number of implicit conjectures.

The results show that out of 18 students (some were on sick leave), 13 achieved a significant learning rate. Only 3 students were under 50% and 1 only could not provide evidence of having reorganized his previous knowledge with the elements of the laboratory.

In particular, the discussions with the students showed that almost all of them would affirm without hesitation that there are straight circumferences; more than half, that the circumference gets shorter if it is "too large", that there is one equilateral and rectangle triangle and that parallel straight lines will meet on large scale. Translating qualitative aspects of the circumference into quantitative, by π and the ratio C/d, proved to be a barrier to all students – except for two. We interpreted it as caused by didactic reasons: it was a new notion just introduced in the "intermezzo lecture" (see Attachment 1) and the concept of ratio had not even completely discussed, because the students had yet to study the difference between ratio and fraction as operator.

The collective sessions showed a large gap between the geometric cognition that students were able to display orally by verbal and non verbal communication and the verbal written test. It was evident that students by the use of their body, examples and mimic could complement their verbal expressions - to communicate anyhow effectively.

The effect of insufficient logico-linguistic competence was consistent with our discussion about the conditions of the observation in paragraph 5.2: students weak on this competence would appear very weak in connecting and integrating their sparse learnings. We have made the conjecture that a weak logico-linguistic competence creates a boundary, a ceiling, to geometric cognition as required at school.

We confirm that the didactic contract deprived our written test of lot of its potential effectiveness. The proof is in the ensuing collective discussion: in it, the students lived a series of problematic

situations and engaged themselves in a problem solving mobilizing all their knowledge and their processing power to get over the issues – in contrast to the written test, in which they spent all their energy trying to guess the right answers.

6.6 The students response: synthesis

We estimated that almost all students:

1. Assimilated and deployed successfully the straight line model, in particular its kinematical aspect - after the initial difficulty inherent in talking about movement and in thinking analytically about it.
2. Were able to apply the straight line model with the perceptive and intuitive model of curvature of a surface to really see straight lines of curved surfaces
3. Their perception, understanding and imagination about their land changed to include straight movements and curvature as well as the associated basic paradoxical aspects

We estimated that more than half of the students:

1. Have changed their conceptions and convictions about triangles, circumferences and parallels to account for their simplest non Euclidean behaviors
2. Their perception, understanding and imagination about their land changed to include triangles that become spherical, circumferences that become straight lines and parallel straight lines that meet

We estimated that about one quarter of the students:

1. Have developed integrated models of straight lines, triangles, circumferences and parallel lines on a curved & flat surface allowing them to interpret non Euclidean deviations in terms of curvature
2. Have realigned to the epistemological model their way to think of their movements and of geometrical figures on the terrestrial surface

We estimated that almost all students arrived at a new image of mathematics as “multidisciplinary” and concerning the “outdoor world”– in which one “can say what he thinks” and “try it out”: what we call the physics of movements.

Chapter 7: Conclusions

7.1 Research questions and hypothesis

With respect to our research questions:

1. What is a set of essential psycho-educational requirements that are most critical for an effective learning of (non Euclidean) geometry in our environment?

We have taken a broad view across different fields and, because of our radical epistemological approach, we have found ourselves questioning some aspects of mainstream theories – indeed, limiting the use that we could make of them. Yet, the foundational theories of reference were enough to arrive at a set of guidelines for the critical choices of our project. The work is in chapter 2 and has attained its aim by providing guidance throughout the entire project.

2. What is a specific epistemology of geometry (Euclidean & non Euclidean), and a didactical transposition, built on the key ideas of Riemannian manifolds and general relativity, that can fulfill the psycho-educational requirements?

We think we provided an answer radical enough to propose a resolution of the dilemma Euclidean versus non Euclidean by a dialectical synthesis, not abstract, but the most earthly possible: our land – and yet, in it are embedded some of the profound concepts of general relativity and Riemannian manifolds. We had to rethink mathematics as physics, and figures as movements – to design an embodiment of geometry embedding both Euclidean and non Euclidean behaviors as they are in general relativity. Chapter 3 and 4 contain this foundational work.

3. What is a laboratory's design based on the above epistemology and transposition and driven by the above requirements?

Grounded in our epistemology and psycho-educational view, we designed from the ground up our laboratory embodying our Riemannian manifolds geometry in our students experiences and adapting it continuously to our learning on the field. Our students completed the course in line with our psycho-pedagogical requirements. Our design concept is in chapter 5, while the handouts in Attachment 1 give a much more concrete idea of the laboratory.

4. What is the response of the students, in broad terms of cognition and meaning, to the implementation of such a laboratory?

Almost all of them have learnt to think kinematically and changed, to different degrees, their conceptions and convictions about triangles, circumferences and parallels to account for their simplest non Euclidean behaviors – both for their land (world) and for the object "sphere". Curvature and "space" became part of the intuitive geometric cognition of most of the students – that were able to use them to explain non Euclidean situations. Students displayed an unexpected easiness about accepting non Euclidean behaviors of their world – suggesting to us the conjecture that "Euclidean cognitive ruptures" might not originate entirely from epistemological (or even ontological) reasons, but might be, at least partially, an effect of our teaching Euclidean geometry instead of non Euclidean Riemannian geometry (embodied as physics of movements).

For our hypothesis of research, we believe them to be confirmed. It is possible to transpose some significant core concepts of Riemannian manifolds and general relativity to re-conceptualize non Euclidean aspects to fulfill our students requirements and bring about tangible concrete changes in students perception and cognition and geometry.

7.2 Value of the research

We believe that students have deepened, extended and made more robust their learning of foundational concepts working on them from sensorimotorial tiers up to logico-linguistic processing and passing through didactical situations positively mobilizing their affective and personal involvements - eventually gaining a new awareness of how amazing their bi-dimensional space may be. Many of them have learnt that our common use of the words and language fails to capture the experience of the world geometry and more flexibility is required with language. They have seen a new mathematics out of the schoolroom and with no numbers, that can tell us "something of what we cannot see".

For our mathematical teachers community, this research might be interesting, for its being the only of its kind and for the reflections it may trigger about some further experiments of a combined Euclidean and non Euclidean transposition of geometry. Moreover, we think it might be useful to help understand geometry and general relativity.

For the teacher, it was not only a challenging opportunity to develop his transposition skills, design and organization skills for a complex laboratory in a highly diversified class, but it was also an enriching experience to learn about our students geometric cognition, kinesthetic intelligence and logico-linguistic skills. To find a solution to the challenging requirements of the student centered pedagogy, the teacher had to be open enough to change from the ground up his conception of

geometry and to reorganize his previously separated conceptions of the many mathematics (including general relativity).

7.3 Many vs. One Geometry

In our opinion, an exclusive focus on the Many Geometries paradigm and the comparative geometry prevents from understanding the revolution of geometry of general relativity (e.g: space has no meaning, space time is curved, its curvature is variable but locally flat, geometry of the world's space has no sense if it is decoupled from the physical features of the objects within it)

The paradigm we have proposed, one geometry of space that can modulate in itself many geometries of figures, helps to see geometry of the world as both Euclidean and non Euclidean. Though non Euclidean geometries might have no relevance to our students, their non Euclidean character is more relevant than ever: it is indissolubly linked to gravity itself (tidal effect).

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Attachment 1: Lectures handouts/ worksheets

Lectures handouts/ worksheets

Schede per gli allievi

UD	Lecture	Topic	Pages	Duration
Entry Assessment				
UD 1	Launch	"The Idea"	p. 3	0.2h
	Lecture 1	Model of straight line	p. 4 - 13	2h
UD 2	Diagnostic Assessment	Earth's Model	p. 15	0.5h
	Lecture 2	Riemannian Curvature	p. 16 - 22	2h
	Lecture 3	Straight lines locally	p. 23 - 25	1h
	Lecture 4	Straight lines: non Euclidean behavior	p. 26 -29	2h
Intermezzo Lecture, Circumference				
	Introduction	Circumferences locally	p. 31 - 40	2h
	Exercises		p. 41 - 42	1h
UD 3	Lecture 5	Circumferences: non Euclidean behaviors	p.44(51) -48(54)	2h
	Lecture 6	Straight lines and 2D maps	p. 49 - 50	1h
UD 4	Lecture 7	Triangles: non Euclidean behaviors	p. 56 - 58	1.5h
UD 5	Lecture 8	Parallel lines: non Euclidean behaviors	p.60(62)-61(63), 64	1.5h
Exit Assessment				

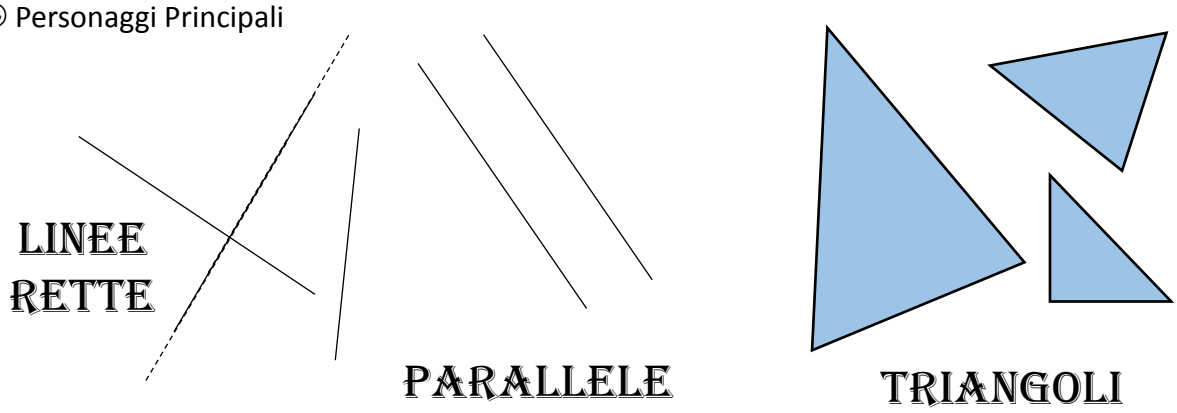
Lavoro di Diploma di M. Banfi
UD 1
Schede per gli allievi

«Geometria Oltre L'Orizzonte»

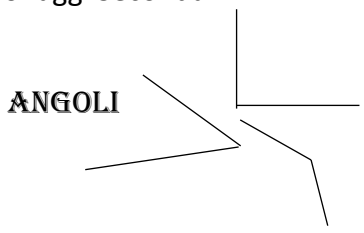
UD1 v1.1
LD M. Banfi

Foglio 0B

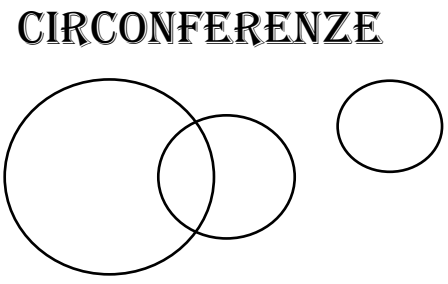
☺ Personaggi Principali



☺ Personaggi Secondari



☺ Personaggi non Confermati



COSA SCOPRIREMO

- Che questi personaggi possono **comportarsi** in modi in e mai visti prima!
- Che fanno così quando, sotto sotto, c'è un **personaggio misterioso**

- Il viaggio durerà diverse lezioni
- Partiremo **dalle radici dei nostri personaggi**, quando sembrano avere poco di matematico e molto di «movimento»

«Geometria Oltre L'Orizzonte»

UD1 v1.1
LD M. Banfi

Foglio 0A

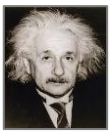
Grandi scienziati e matematici del nostro percorso

Euclide (ca. 300 A.C.) - Studiò alla perfezione i «comportamenti normali» dei personaggi e scrisse la geometria della SM e di parte della SMS



Circa 2000 anni più tardi, tra il 1820 e il 1840, Friedrich J. C. Gauss, János Bolyai a Nikolai I. Lobachevsky, indipendentemente l'uno dall'altro, scoprirono che i nostri personaggi potevano comportarsi in modi inaspettati e rivoluzionari!

Bernard Riemann – attorno al 1855, scoprì e studiò il «personaggio misterioso» che fa cambiare incredibilmente i comportamenti dei personaggi



Albert Einstein – attorno al 1915, usando questi «incredibili comportamenti» e il «personaggio misterioso» spiegò come funziona l'universo e scrisse le equazioni che predissero i «buchi neri»

Attività: l'idea di Andar Dritto e la Retta

Attività A

Discutete esempi di oggetti che vanno dritti (potete usare i materiali)

Scegliete un esempio e spiegatelo a fondo ai compagni

Attività B

Prendete la bicicletta con la freccia (potete bagnare le ruote e passare sulla passatoia di carta)

Rispondete:

1. Cosa sentite se avete gli occhi chiusi e siete **sulla** bici quando va dritto? E quando curva?
   .....
2. Cosa vedete se avete gli occhi aperti - nei due casi?
   .....
3. Che **tipo di traccia** lascia la bici nei due casi?
   .....
4. Che relazione c'è tra la freccia e la traccia?
   .....
5. Che differenza c'è tra queste tracce e le linee rette della geometria?
   .....

Mettete tutto insieme e spiegate ai compagni

Attività: l'idea di Andar Dritto e la Retta

Attività C

Prendete uno dei rotoli di carta

Rispondete:

1. Che tipo di traccia lascia srotolandosi?
   .....
2. Se cercate di srotolarlo facendolo curvare, si riesce a tenere il nastro a terra? Cosa «sente» il nastro?
   .....

Mettete tutto insieme e spiegate ai compagni

Attività D

Sul campo da basket due allievi si mettono distanti l'uno dall'altro

Rispondete:

1. Quanti percorsi diversi può fare una formica in bicicletta per andare da un allievo all'altro?
   .....
2. E se può andare solo dritta?
   .....
3. Cosa indica la freccia?
   .....
4. In quanti modi potete srotolare il nastro da un allievo all'altro – senza curvarlo?
   .....
5. Quante linee rette ci sono che congiungono i due allievi?
   .....

Mettete tutto insieme e spiegate ai compagni

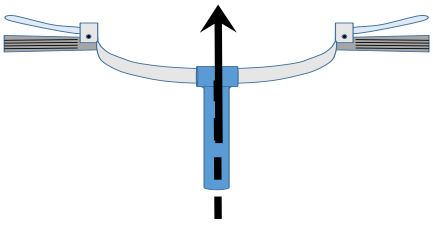
Risultati: l'idea di Andar Dritto e la Retta

ANDAR DRITTO:
Come lo si capisce stando sulla bicicletta?

SE VA DRITTA (sterzo bloccato) …

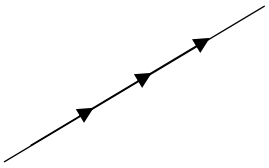
COSA SENTO
Che sto in centro in equilibrio, non sbando, non m'inclino

COSA VEDO
La freccia è fissa, allineata con la canna



CHE TRACCIA LASCIO
Dietro di me vedo **<u>una retta</u>**

RELAZIONE TRA FRECCIA E TRACCIA
La freccia «scorre dentro» la traccia

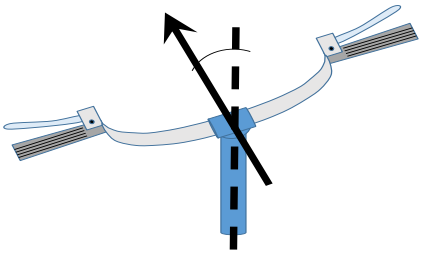


La TRACCIA è un modello della LINEA RETTA della geometria

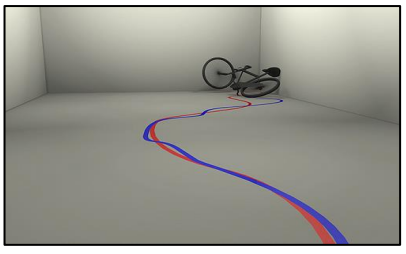
SE CURVA (sterzo libero) …

COSA SENTO
Che non sto in centro, sbando, m'inclino

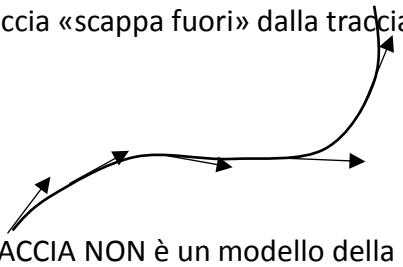
COSA VEDO
La freccia gira, è disallineata con la canna



CHE TRACCIA LASCIO
Dietro di me vedo una linea non retta

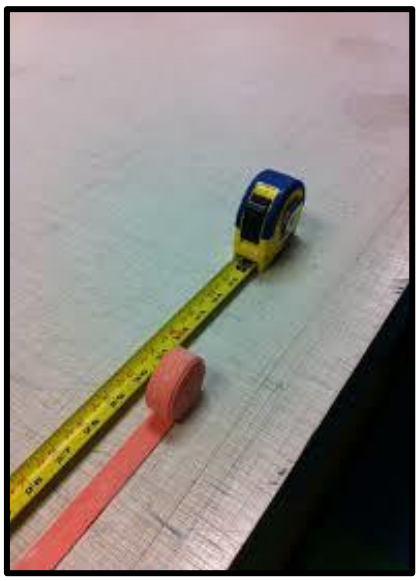


RELAZIONE TRA FRECCIA E TRACCIA
La freccia «scappa fuori» dalla traccia



La TRACCIA NON è un modello della LINEA RETTA della geometria

Risultati: l'idea di Andar Dritto e la Retta



Un rotolo che si svolge va sempre dritto.

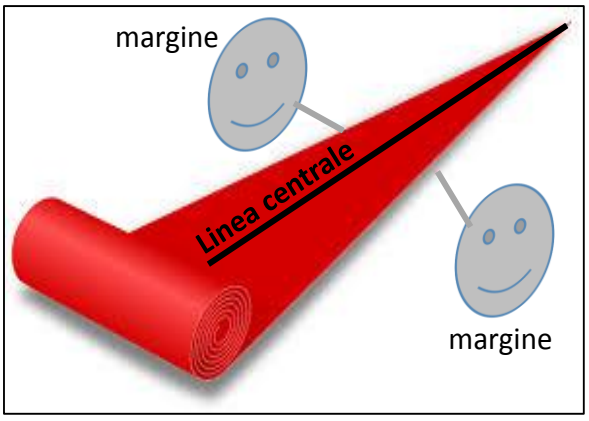
Il nastro è un'immagine della linea retta, **se non guardiamo la larghezza**

ANDAR DRITTO …

Cosa sentono i margini del rotolo di carta?

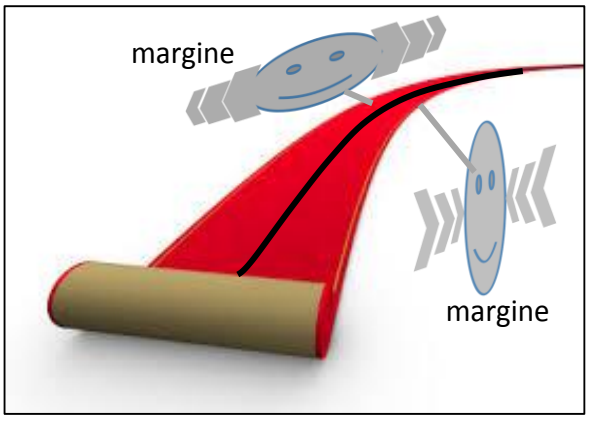
SE VA DRITTO …

I bordi sono rilassati, la linea centrale è perfettamente aderente



SE CURVA …

Un margine è compresso e l'altro è stirato; la linea centrale può essere sollevata; un margine può essere sollevato e l'altro no

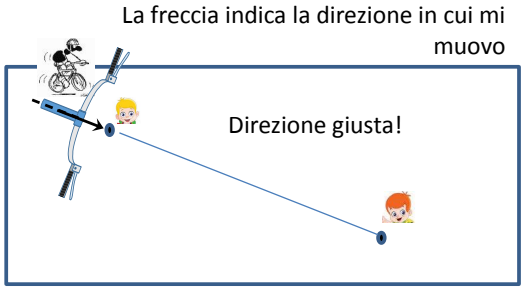


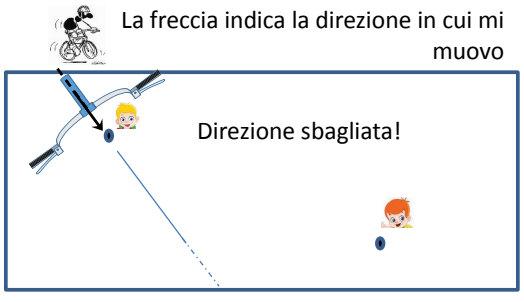
Risultati: l'idea di Andar Dritto e la Retta

SE VA DRITTA (sterzo bloccato) …

SE CURVA (sterzo libero) …

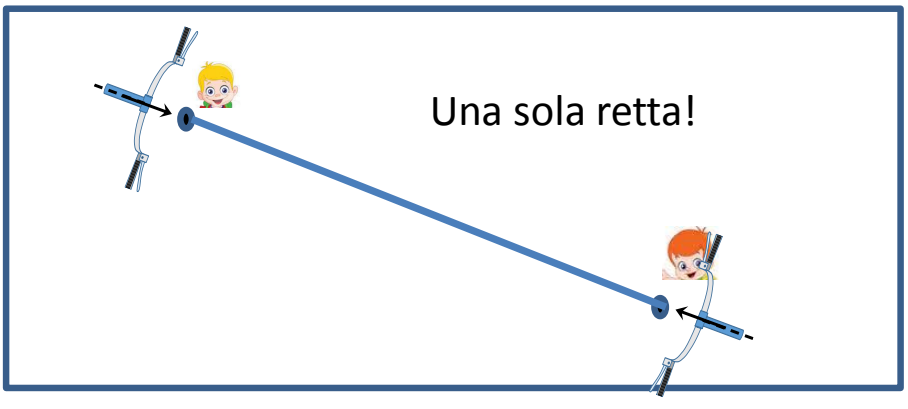
QUANDO ANDIAMO DRITTO:





CONCLUSIONE

C'e' una sola retta che congiunge i due allievi



SINTESI: Idea di Andar Dritto e Retta

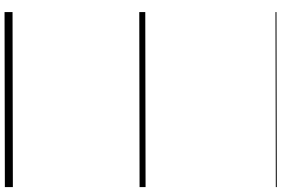
Possiamo prendere come modello di LINEA RETTA
la TRACCIA lasciata da una bicicletta che VA DRITTA perfettamente

IN MATEMATICA:

Dobbiamo immaginare che la **LINEA RETTA**

1. abbia solo LUNGHEZZA
2. non abbia LARGHEZZA

E' come se fosse un filo più sottile di qualsiasi filo



Retta r

Si dice quindi che la retta è MONODIMENSIONALE (mono=uno; una dimensione)

Dobbiamo immaginare che il **PUNTO**

1. non abbia ne' LUNGHEZZA ne' LARGHEZZA

E' come la capocchia di uno spillo più sottile di qualsiasi spillo

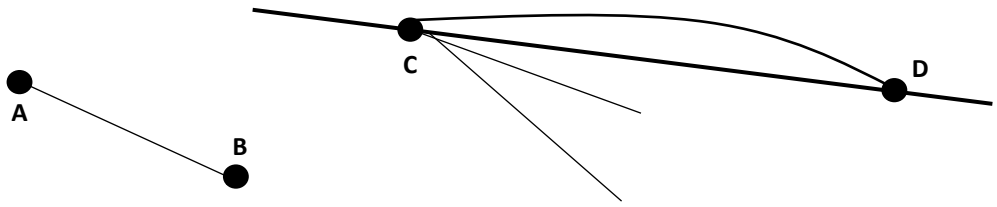
Punto P

Si dice quindi che il punto è ZERODIMENSIONALE (zero dimensioni)

Rifletti: Cos'ha e cosa non ha una SUPERFICIE? ……
……

SINTESI: Comportamenti delle Rette

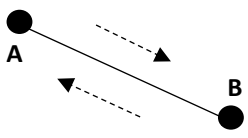
COMPORTAMENTO DELLE LINEE RETTE:
Per due punti distinti passa SEMPRE solo una retta



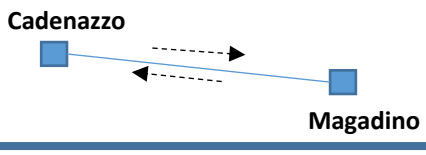
Possiamo scegliere i punti in tutti i modi possibili, ma di rette ne passerà sempre una sola!

ATTENZIONE:

Il modello della retta non è il «movimento», ma solo «**la sua traccia».**
Che io vada in una direzione o nell'altra **la traccia,** e quindi «**la retta»,** è sempre una!



PER ESEMPIO:
Tra Quinto e Cadenazzo c'è solo UN binario - che posso percorrere in due direzioni



Attenzione: **retta, semiretta e segmento sono tre figure diverse.** Quando non è importante distinguere di quale delle tre si tratti, noi useremo semplicemente la parola «retta».

Segmento AB Semiretta s con origine in P Retta r

«In Pillole»: Idea di Andar Dritto e Retta

Linee rette

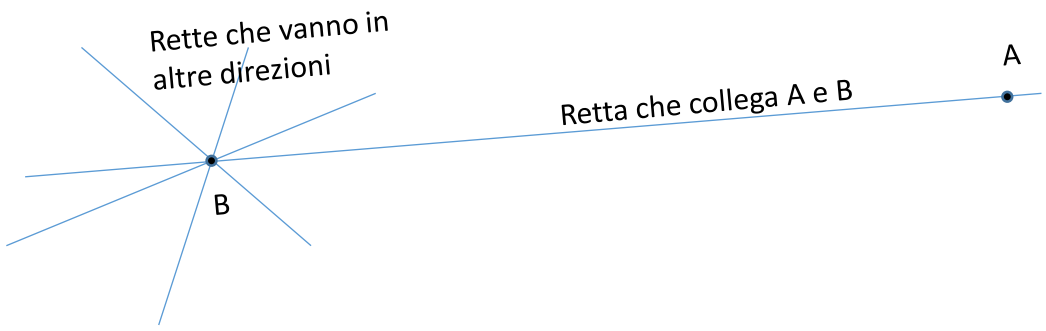
- **Le tracce di larghezza zero** di una bicicletta che va dritta sono il nostro modello di linea retta
- **C'è solo una retta che collega due punti**, perché c'è solo una strada che la bici può fare andando dritto

Direzione

- E' la linea immaginaria del tuo sguardo quando vai dritto
- La freccia sulla bici indica la direzione

Come trovare la linea retta

- Scelgo in che direzione m'interessa andare e parto con la bicicletta: senza curvare, sbandare, stando al centro e senza far ruotare la freccia sul manubrio. La traccia è la linea retta nella direzione scelta
- Oppure srotolo un rotolo di carta o di scotch nella direzione scelta. La traccia è la linea retta nella direzione scelta



La bicicletta matematica

E’ una bicicletta immaginaria con la freccia sul manubrio, che:

1. Va sempre solo dritto
2. Lascia una traccia monodimensionale
3. Davanti a lei le montagne e i mari si spianano e tutto viene perfettamente asfaltato
4. Va a qualsiasi velocità desiderata

Lavoro di Diploma di M. Banfi
UD 2
Schede per gli allievi

Pietro e la Terra Rotonda (formativa)

UD2 v1.1
LD M. Banfi

Foglio 0

Pietro fa le Elementari nell'alta Valle Bianca. Non ha idea di cosa voglia dire che la Terra è rotonda. Non ha mai visto la pianura o il mare. E' sempre rimasto tra le sue montagne. Ha chiesto aiuto a voi per capirci un po' su questa cosa della Terra rotonda!

Potete usare i materiali che volete, ma preparatevi anche a rispondere a queste domande:

«Le foto della Terra mostrano un cerchio non una sfera!»

«Sono stato ad Arosio: si vede la pianura padana ed e' piatta!»

«Come potrebbe essere rotonda se le montagne e i continenti galleggiano sul mare molto profondo?»

«La mia amica talpa mi ha detto che lei scava delle gallerie perfettamente dritte esattamente come sua cugina che vive in un pianeta a forma di cubo!»

☺ ☺

Attività: Usciamo dal Foglio

Partenza

Immaginate di partire da un punto del vostro foglio o del campo da basket e con due bici matematiche di tracciare due semirette.

Discutete: cosa fate? Cosa vedete? Cosa sentite? Che problemi ci sono?

Primo Ostacolo

Montagne, valli, case, eccetera, bloccano subito le nostre biciclette. Allora, immaginiamo che il paesaggio sia perfettamente spianato e asfaltato (sarà sempre così per le bici matematiche). Le immagini proiettate possono aiutare la vostra immaginazione.

Discutete:

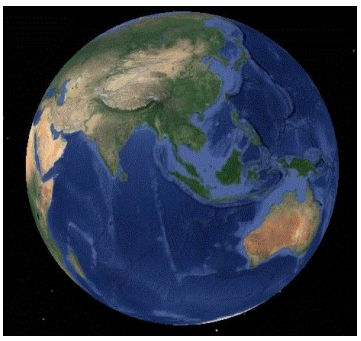
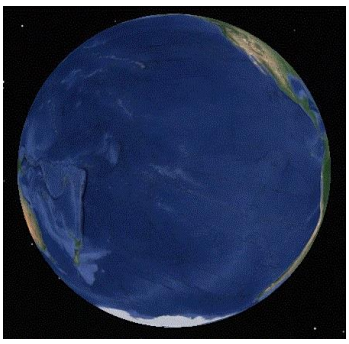
1. stando in piedi cosa vedete e come vi sentite su questa superficie spianata e asfaltata?
2. adesso partite con le bici: cosa fate? Cosa vedete? Cosa sentite? Come si comportano le semirette che lascia la bicicletta?

Secondo Ostacolo

Il fatto che la Terra è «rotonda» non blocca il nostro esperimento. Ma dobbiamo fare una lunga deviazione, perché «sotto» la parola «rotonda» si nasconde … il «personaggio misterioso»!

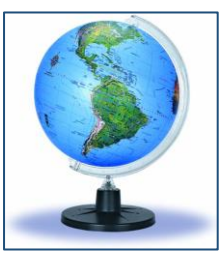
Attività: Deviazione: La Terra 3D

Guardate le foto del pianeta Terra da tre diverse posizioni.

Guardate l'animazione Google in zoom out e poi in zoom in di Gravesano

Osservate e toccate il globo terrestre geografico e la sfera di plastica.
Sono modelli 3D della Terra (3D vuol dire che sono solidi e non superfici - come le carte geografiche, che sono 2D)




Discutete:

1. la sfera di plastica vi dà informazioni sulla geografia fisica della Terra? .....
2. E sulla geografia politica? .....
3. Che cos'ha in comune con la Terra? .....

Rispondete/ completate:

1. per vedere coi vostri occhi che la Terra ha la forma di una sfera, dove dovreste andare? .....
2. la forma della Terra può essere modellata con un solido: la .....

Sintesi: Modello 3D della Terra

Da Ricordare

La Terra è un solido. La sua forma è approssimativamente quella di una sfera. La sfera sarà il nostro modello 3D della Terra.

Terra

Modello 3D →

Sfera

Per Approfondire

La forma della Terra non è esattamente una sfera perché:
1. è leggermente schiacciata ai poli (ellissoide)
2. è irregolare (montagne, valli, eccetera) (geoide)

Cerca informazioni su questi altri modelli (ellissoide e geoidi) della Terra.

Le dimensioni della Terra, dipendono dal modello 3D che si sceglie (sfera, ellissoide o geoide).

Cerca informazioni sulle dimensioni della Terra nel caso del modello sferico.

Attività: Deviazione: La Superficie della Terra

“Tra una talpa e un gatto, chi vive la Terra come un solido e chi come una superficie?” ……..

E tu, come la pensi per tutti i tuoi movimenti? ……..

Osservate il globo terrestre geografico, la sfera di plastica, la calotta sferica, il piano del banco e la pagina di copertina del libro





- Prendete tra indice e medio la calotta di plastica.

Rispondete: che parola usereste per descrivere quello che avete tra le dita: solido o superficie? ……

- Fate scorrere delicatamente il palmo della mano sulla superficie di questi oggetti

Rispondete: è più preciso dire “sento la superficie dell’oggetto” o dire “sento l’oggetto”? ……

- Fate scorrere ancora delicatamente il palmo della mano su queste superfici, ma adesso ad occhi chiusi

Rispondete: per quali superfici usereste le parole “curvo/ curvatura” e “piatto?
…….

- Lavorate con tutto il corpo con le Swiss Balls

Rispondete: stando di pancia sulla Swiss Ball sentite la curvatura? E se fosse grane 100 km, la sentireste?.....

Chi deve fare i conti con la curvatura della superficie terrestre: il gatto o la talpa? ……..

Sintesi: Modello della Superficie della Terra

Da Ricordare

Abbiamo scoperto con il palmo della mano l'idea di curvatura di una superficie (sferica , nel nostro caso)

Le superfici della sfera e della Terra sono ……
Noi viviamo su una superficie ……..
Un piano o la superficie di un banco sono ……..

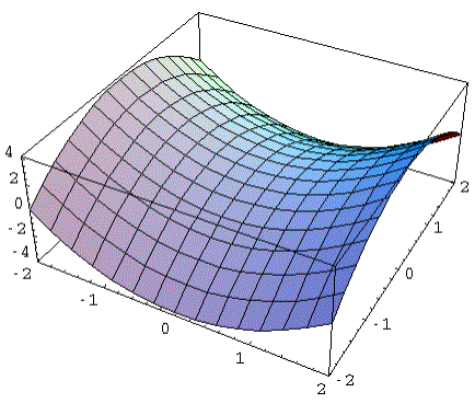
Per Approfondire

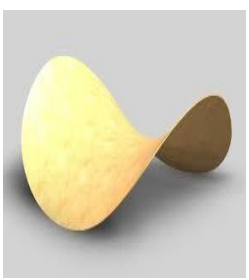
Che differenza c'è tra la curvatura degli oggetti visti in classe e quella di:

1. un cilindro (puoi prendere un rotolino di carta o una lattina);

2. un uovo (attenzione);

3. la "sella" e il "pringle" - riportati nella figura qui sotto?





Attività: Deviazione: Superficie curva ma piatta?

Abbiamo detto che noi viviamo su una superficie curva, ma se immaginate di essere in un grande campo – come quello della foto

vi sentite su una superficie curva o piatta?
…… come si spiega?????

Ad occhi chiusi appoggiate il palmo della mano prima sulla superficie della sfera di plastica e poi sulla swissball

Rispondete:

1. come cambia la sensazione della curvatura passando dalla sfera di plastica alla swissball?.....
2. se la swissball fosse grande 1km, sentireste ancora la curvatura con la mano? ……potreste almeno vederla?.....
3. e se fosse 1000 km, vedreste che curva? ……
4. potete immaginarvi che star sulla superficie terrestre sia come stare sulla superficie di una swissball grandissima? ……quanto grande?.....
5. quale è quindi il motivo per cui ci sembra di vivere su una superficie piatta? ……..

Verificate (guardate e sentite con un polpastrello) sul globo la curvatura della superficie del Sottoceneri

Guardate il filmato del veliero e dell'orizzonte

Rispondete: se NON ci si allontana più di ……. Km è come se non ci fosse curvatura

Discutete: cosa cambierebbe se una mattina vi svegliaste e scopriste che il vostro corpo si è allungato e i vostri piedi sono a 5000 km dalla testa?

Sintesi: Superficie curva ma piatta?

Da Ricordare

1. la superficie terrestre pur essendo curva ci appare piatta perché viviamo in una zona che è piccolissima rispetto all'intera superficie.

2. questo non vale solo per la Terra, ma per tutte le sfere e (approfondimento) per moltissime superficie curve.

Da Fare Attenzione

Spesso si usa la stessa parola per due cose diverse. Infatti:

Se diciamo linea "curva", intendiamo che il pilota della bici ha curvato – se non lo fa, abbiamo una linea retta

Se diciamo superficie "curva", significa che facendoci scorrere sopra il palmo della mano (magari una mano gigante) sentiamo che non è piatta – diciamo «curva» ma non significa che fa curvare una bici che ci corre sopra (torneremo su questo punto più avanti)

Per approfondire

Verifica che anche per il cilindro, l'uovo, la sella e il pringle vale la proprietà che la superficie appare piatta in una zona sufficientemente piccola rispetto all'intera superficie

Cerca informazioni su come avevano fatto i Greci a capire che la superficie terrestre è curva

Attività: Rette su una Superficie Piatta e Limitata

Possiamo adesso tornare al nostro problema!

Immaginate di partire da un punto del vostro foglio o del campo da basket e con due biciclette matematiche di tracciare due semirette e immaginate che il paesaggio sia perfettamente spianato e asfaltato.

Rispondete: se non andiamo più lontani di 5 km, come ci appare la superficie terrestre? ……..

Allora cominciamo a vedere cosa succede stando dentro un recinto immaginario a una distanza di 5 km

Nella pagina successiva trovi una lista di comportamenti delle rette da investigare

Attività: Rette su una Superficie Piatta e Limitata

Discutete i seguenti comportamenti delle bici matematiche e rette, indicando V (vero) o F (falso) (c'è un recinto a 5 km)

Moto Rettilineo

- ❑ La bici può andare dritta senza limite
- ❑ La bici che va dritta si allontana sempre di più dal punto di partenza
- ❑ La bici che va dritta non ritorna mai al punto di partenza
- ❑ Due bici che partono da uno stesso punto e vanno dritte in diverse direzioni si allontanano sempre più l'una dall'altra
- ❑ Per andare dritto in bici da un punto ad un altro c'è sempre una sola direzione
- ❑ Quella dritta è sempre la strada più corta tra due punti

Geometria

- ❑ Un segmento può essere prolungato in modo rettilineo senza limiti
- ❑ E la lunghezza del segmento aumenta senza limite
- ❑ Non si può ottenere una linea chiusa prolungando in modo rettilineo un segmento
- ❑ Prolungando in modo rettilineo segmenti uscenti da uno stesso punto i loro estremi si allontanano
- ❑ Presi due punti qualsiasi c'è un solo segmento che li congiunge
- ❑ Il segmento ha la proprietà di essere la linea più corta tra due punti

Quiz

- ❑ Se Elena è arrabbiata con Marco e se ne va via dritta, sarà sempre più lontana da Marco
- ❑ Se tre amici partono all'avventura da uno stesso punto andandosene dritti in tre direzioni diverse non si vedranno più
- ❑ Se Elena dice a Marco "me ne vado il più lontano possibile da questo posto", Marco non potrà mai raggiungerla se lei non gli dice dov'è andata
- ❑ Se Sven e Paolo fanno una strada tutta dritta per andare da Monica, allora hanno fatto la strada più corta

Il nostro mondo

- ❑ Se volo dritto da un posto ad un altro sono sicuro d'aver fatto la strada più breve
- ❑ La più corta rotta aerea che collega due posti è sempre una sola
- ❑ Se vai dritto sul terreno, sulla cartina geografica la tua traccia è sempre un segmento di retta

Sintesi: Rette su una Superficie Piatta e Limitata

Da Ricordare

1. Le rette si comportano nello stesso modo su un foglio di 20 cm o su una parte della superficie terrestre ampia qualche chilometro

2. Quello che conta è che la superficie “su cui recitano” (il loro palco!) sia piatta e limitata, come, per esempio: una lavagna oppure una zona sufficientemente piccola, e ben recintata, di una qualsiasi sfera

3. Quindi, il foglio di carta è un buon modello della superficie terrestre se stiamo in una zona piccola qualche chilometro

Per approfondire

Rifletti se questi fatti valgono anche per il cilindro, l’uovo, la sella e il pringle

Attività: Rette sulla Superficie Sferica

Adesso togliamo il recinto: le bici possono scorrazzare per tutta la superficie terrestre

Discutete: pensando al veliero, alla superficie che s'incurva – il foglio di carta piatto vi sembra un buon modello? Come lo vorreste cambiare?

La Sfera di Lénárt

- Immaginate di andare dritti sulla superficie terrestre (con la bici).

Discutete:

1. quale linea ottenete sulla sfera? (ricordatevi quali sensazioni si hanno andando dritti e cosa fa la freccia sul manubrio)
2. il nastro biadesivo conferma le vostre idee?
3. cosa avreste bisogno per tracciare le rette sulla sfera in modo più preciso?

Rette o curve?

- Completate:

1. Quello che vediamo sulla sfera è quello che vedremmo dallo ……
2. le rette ottenute sono proprio i prolungamenti dei segmenti del foglio e del campo di ……
3. siccome il pilota della bici non ha "sentito niente" e non ha girato la freccia sul manubrio queste linee sono …….
4. è la superficie su cui vivono che ……. (non confondere la curvatura della superficie con quella della traccia della bici)

Completate:

1. Chi vive sulla superficie le vede come .....; chi vive nello «spazio sotto la superficie», la talpa, o sopra la superficie, un satellite spaziale, le vede come …..
2. le linee ottenute sono rette della …… ; non sono rette dello …….

Attività: Rette sulla Superficie Sferica

Per esercitarvi/ giocare

Rispondete:

1. la talpa deve andare dal polo sud al polo nord, che strada fa? Passa per ……

2. il gatto deve andare dal polo nord al polo sud, che strada fa? Va dritto stando sulla ……

3. cosa dice la talpa della strada che ha fatto il gatto? Che non è …… ma è ……

Il gioco del Test Psicomatecinestetico ☺

1. Disegna un segmento di retta su un foglio. Adesso incurva leggermente il foglio come se tu volessi farne un cilindro.
2. Scegli una risposta:

A. Il segmento è diventato curvo
B. il segmento non è cambiato, è ancora dritto, è cambiata la curvatura della superficie

«Chi sei»
Se hai scelto A, sei una creatura tridimensionale: hai le idee chiare ma tendi a vedere le cose solo dal tuo punto di vista!
Se hai scelto B, sei una creatura tridimensionale (che dallo spazio vede la superficie incurvarsi) ma anche bidimensionale (che stando dentro nella superficie vede il segmento non fare nulla): riesci a vedere il punto di vista degli altri, ma a volte manchi di chiarezza!

Attività: Rette sulla Superficie Sferica

Rispondete: quali sono le differenze più grosse tra il comportamento della retta su una superficie piatta (limitata o non) e il comportamento sulla superfice sferica?

…….

Aiutandovi con la sfera di Lénárt, discutete cosa fanno le bici matematiche e le rette, indicando V (vero) o F (falso):

Moto Rettilineo

- ❑ La bici può andare dritta senza limite
- ❑ La bici che va dritta si allontana sempre di più dal punto di partenza
- ❑ La bici che va dritta non ritorna mai al punto di partenza
- ❑ Due bici che partono da uno stesso punto e vanno dritte in diverse direzioni si allontanano sempre più l'una dall'altra
- ❑ Per andare dritto in bici da un punto ad un altro c'è sempre una sola direzione
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- ❑ Prolungando in modo rettilineo segmenti uscenti da uno stesso punto i loro estremi si allontanano
- ❑ Presi due punti qualsiasi c'è un solo segmento che li congiunge
- ❑ Il segmento ha la proprietà di essere la linea più corta tra due punti

Quiz

- ❑ Se Elena è arrabbiata con Marco e se ne va via dritta, sarà sempre più lontana da Marco
- ❑ Se tre amici partono all'avventura da uno stesso punto andandosene dritti in tre direzioni diverse non si vedranno più
- ❑ Se Elena dice a Marco "me ne vado il più lontano possibile da questo posto", Marco non potrà mai raggiungerla se lei non gli dice dov'è andata
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Il nostro mondo

- ❑ Se volo dritto da un posto ad un altro sono sicuro d'aver fatto la strada più breve
- ❑ La più corta rotta aerea che collega due posti è sempre una sola
- ❑ Se vai dritto sul terreno, sulla cartina geografica la tua traccia è sempre un segmento di retta

Sintesi: Rette sulla Superficie Sferica

Da Ricordare

La retta è la traccia della bici matematica

Se andiamo dritti su una superficie sferica non sentiamo se la superficie è curva o piatta (tutti i giorni succede proprio questo)

Sulla superficie sferica le rette si comportano in modo sorprendente: si richiudono su stesse, sono tutte isometriche, eccetera

Senza andare nello spazio o dentro la Terra posso scoprire che la superficie è curva e sferica grazie a questi comportamenti delle rette

E' la curvatura della superficie che crea questi comportamenti non ordinari – il personaggio misterioso

Per approfondire

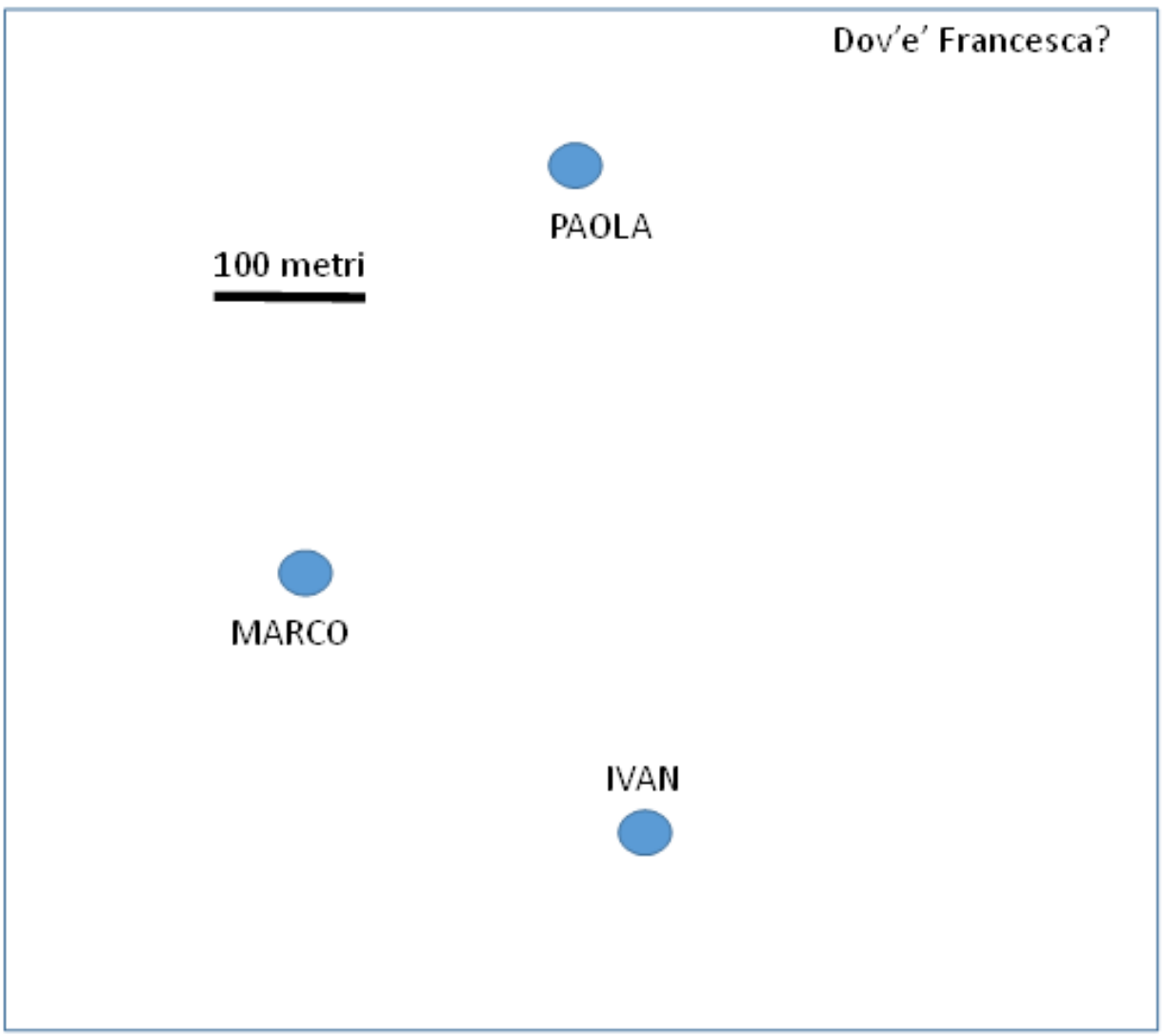
Rifletti se questi fatti valgono anche per il cilindro, l'uovo, la sella e il pringle

Lavoro di Diploma di M. Banfi
Intermezzo Lecture: Circonferenza
Schede per gli allievi

Distanze e circonferenze

Francesca si trova a 100 m da Marco, a 200 m da Ivan e a 300 m da Paola.

Come trovare la posizione di Francesca sulla mappa in scala qui sotto con una strategia che si possa applicare a colpo sicuro anche cambiando le distanze e le posizioni dei suoi amici? .... ***Scopriamo la geometria che ci aiuterà!***



Attività sugli oggetti reali, sulle *c* _ _ _ _ _ _ _ _ _ _ _ *e* e sui *c* _ _ _ _ *i*

1 Discutete in gruppo dove si possono vedere circonferenze o cerchi negli oggetti a disposizione.

2 Mettete in comune con il resto delle classe le vostre osservazioni

3 In gruppo rispondete o completate:

- Le circonferenze e i cerchi della geometria sono figure ..... che non esistono nella realtà
- Gli oggetti reali sono solo ..... circonferenze o cerchi
- Un .....fatto con un filo sottilissimo ci dà l'idea di una .....
- Un ..... fatto con una lamina sottilissima ci dà l'idea di un .....

Quindi:

CIRCONFERENZA -> è una .... -> che ha una .....-> che si misura per esempio in .....

CERCHIO - > è una .....-> che ha un'.....-> che si misura per esempio in .....

Attenzione: Quando parliamo di "cerchio" intendiamo anche il suo "contorno" (la linea nera – che, in effetti, è una circonferenza)

Attività sulla proprietà fondamentale dei punti della *c* _ _ _ _ _ _ _ _ _ _ _ *a*

Prima Attività

1 In due gruppi, fissate bene tese a terra le cordicelle e disponetevi ai loro estremi.

Discutete come mai un gruppo di allievi appare disposto lungo una circonferenza e l'altro no

2 Mettete in comune con il resto della classe le vostre osservazioni

3 In gruppo rispondete o completate

- Per avere una circonferenza tutti gli allievi devono trovarsi alla stessa ..... dal centro
- Gli allievi possiamo immaginarli come ..... di una circonferenza
- L'allievo al centro come il ..... della circonferenza
- Le cordicelle (segmenti) come ..... della circonferenza
- Attenzione: si usa anche dire, "il" raggio è 2 cm.

Seconda Attività

1 Sul foglio A3 scegliete un punto O e disegnategli intorno due linee chiuse nel seguente modo:

- per la prima – tenete il nastro ben teso e usate un pennarello rosso
- per la seconda – non tenete il nastro teso, scegliete un percorso a piacere e usate un pennarello blu

Discutete che proprietà hanno i punti della linea quando il nastro è teso

2 Mettete in comune con il resto delle classe le vostre osservazioni

3 In gruppo rispondete o completate

- Se il nastro è teso, la linea rossa è una ..... perché la distanza dal punto O è sempre la ..... ; questa distanza è il raggio della circonferenza
- Se il nastro è lasco, la linea blu non è una circonferenza perché la distanza dal punto O .....
- Con una matita colorate in blu la superficie racchiusa dalla circonferenza: il cerchio è l'insieme dei punti di colore ..... e di colore .....

Approfondimento

- Se il nastro è lasco, la linea non è una circonferenza ma è parte del precedente ..... di centro O, perché la distanza da O è minore del .....
- Quindi, tutti i punti del cerchio di centro O hanno la proprietà che la loro distanza da O è ..... o ..... al suo raggio

Terza Attività (Opzionale/ Riserva)

1 Scegliete un punto O. Prima segnate tutti i punti che distano 3 cm da O; poi quelli che distano 5 cm. Scegliete infine un punto P. Discutete se ci sono punti che distano 3 cm da P e 5 cm da O. E se cambiate la posizione di P?

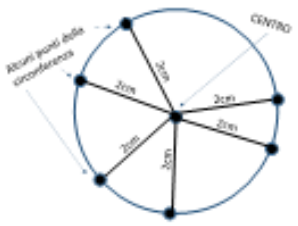
2 Mettete in comune con il resto delle classe le vostre osservazioni

3 In gruppo rispondete o completate

- L'insieme dei punti che sono a distanza r da un dato punto O è la ....., di centro O e raggio r
- Se un punto sta su due circonferenze, le sue distanze dai loro centri sono uguali ai rispettivi .....

Sintesi: Circonferenze e Cerchi

1. Proprietà fondamentale della circonferenza di centro O e raggio r:



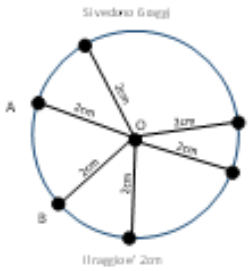
"Tutti i punti della circonferenza si trovano alla ..... distanza dal ..... Questa distanza è il ..... ."

2. Differenza circonferenza/ cerchio

La ....., è una linea, ha una lunghezza e si misura, per esempio, in cm

Il ..... è una superficie, ha un'area (più correttamente, estensione) e si misura, per esempio, in cm^2

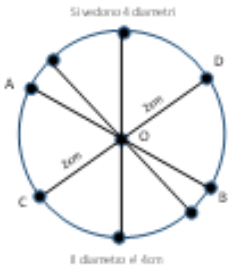
3. La parola "raggio" ha due significati



Indica uno qualsiasi dei segmenti che hanno per estremi il centro e uno dei punti della ....., per esempio: OA, oppure OB, ecc.

Oppure indica la lunghezza di questi segmenti, per esempio: 2cm

4. La parola "diametro" ha due significati

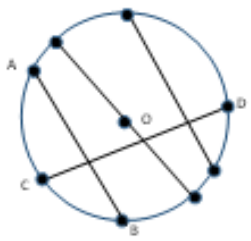


Indica uno qualsiasi dei segmenti che passano per il ....., ed hanno per estremi due punti della circonferenza, per esempio: AB, oppure CD, ecc.

Oppure indica la lunghezza di questi segmenti, per esempio: 4cm

5.Il diametro è il doppio del raggio

6. La parola "corda"



Indica uno qualsiasi dei segmenti che hanno per estremi due punti della circonferenza, per esempio: AB, oppure CD, ecc.

Quindi, i diametri sono particolare corde: **corde di lunghezza** …….

Come la Geometria ci aiuta ad essere giusti nelle gare di atletica: una pista un po' particolare

Mentre in tutto il mondo le competizioni di atletica per i 100 metri utilizzano una pista dritta, la SM di Girotondo ha deciso di fare qualcosa di originale e ha fatto costruire un bella pista circolare con quattro corsie.

Adesso, però è saltato fuori un problema.

Un allievo di prima, Jan, ha fatto notare che se è vero che chi fa un giro sulla corsia più interna corre 100 metri, chi fa un giro sulle altre corsie fa di più di 100 m. Jan vorrebbe misurare con delle corde pezzo pezzo la lunghezza delle altre corsie per trovare quanto sono più lunghe.

Lisa, un'allieva di seconda, molto sveglia, ha detto che non ce n'è bisogno: sapendo che la larghezza delle corsie è 80 cm, con un tre calcoli si sa tutto subito!

Jan non capisce cos'ha in mente Lisa e ha chiesto alla 2A di Gravesano di aiutarlo!

1. *Come spieghiamo a Jan il ragionamento di Lisa?*
2. *Come dovremmo far partire i corridori per essere giusti (equi)?*
3. *Procurati le misure della pista di atletica internazionale (quella da 400 m.) e trova la differenza di lunghezza tra la corsia più esterna e la più interna*

Scopriamo la geometria che ci aiuterà! 😊

Attività sulla relazione tra …… e …….

Prima Attività

1 Per i vostri oggetti, misurate e calcolate **quante volte la circonferenza è più lunga del diametro** e riportate misure e risultati nella tabella qui sotto.

Oggetto	Lunghezza Circonferenza Misurata (C)		=	Diametro Misurato (d)		•	*"Numero di volte"*		C : d
		cm	=		cm	•			
		cm	=		cm	•			
		cm	=		cm	•			
		cm	=		cm	•			
		cm	=		cm	•			
		cm	=		cm	•			

2 Chiedete al maestro il foglio per pubblicare il vostro risultato, completatelo e restituitelo.

3 Preparatevi alla discussione di classe, rispondendo:

- Il "numero di volte" l'avete calcolato dividendo ……. per ……., quindi possiamo indicarlo con "C : d"
- Invece che C : d possiamo scrivere …… e considerarlo come "rapporto" tra la lunghezza della …… e il diametro

- "C : d" varia di poco o di tanto? ……
- E se lo confrontate con quanto varia la circonferenza, "C : d" varia di poco o di tanto? ……
……

4 Mettete in comune con il resto delle classe le vostre osservazioni

5 In gruppo rispondete o completate

Se pensiamo agli errori di misura, e al fatto che nessuna circonferenza reale è matematicamente perfetta, secondo voi, potrebbe essere che C/ d sia costante per le circonferenze matematicamente perfette? .....

In matematica si dimostra che è proprio così:

C/d è costante per tutte le circonferenze (matematicamente perfette)

Il suo valore è indicato con la lettera **π** che si legge **"pi-greco"** (vedi anche scheda di sintesi)

Premi il tasto π della calcolatrice e copia qui il risultato:.....

Breve storia del calcolo di pi-greco e cenno ai numeri irrazionali

GLI ANTICHI

Da quando l'uomo ha inventato la ruota ha sicuramente intuito che C/ d è costante. Circa nel 2000 a.C. i Babilonesi per π usavano l'espressione (3 + 1/8) che è uguale a 3,.....
Gli Egizi usavano (3 + 13/81), che arrotondata al centesimo è uguale a 3,.....

ARCHIMEDE

I Greci usavano un'espressione più precisa - calcolata dal grande matematico e scienziato Archimede (vissuto attorno al 250 a.C.). Essa era (3 +1/7), che arrotondata al centesimo è uguale a 3,..... (se vuoi, puoi scoprire il metodo di Archimede nella scheda successiva)

COMPUTERS

Oggi, con i super computer, conosciamo miliardi di cifre decimali di π; ecco le prime 50:
π= 3,14159265358979323846264338327950288419716939937510 ...

SENZA FINE

Ma quando finiscono queste cifre? O almeno, quando cominciano a diventare periodiche?
Nel 1761, un matematico di nome Lambert, dimostrò quello che molti matematici sospettavano: **le cifre non finiranno mai e neanche diventeranno periodiche!**

IRRAZIONALI

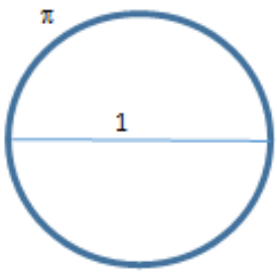
Numeri di questo tipo si chiamano "irrazionali". Oltre a π, ne hai incontrati altri, per esempio: .....

RAZIONALI

I numeri "normali", quelli che hanno solo alcune cifre decimali dopo la virgola (per esempio: .....) oppure che ne hanno infinite ma che ripetono sempre lo stesso gruppo (per esempio: .....) si chiamano "razionali" ed hanno la proprietà fondamentale che **si possono sempre scrivere esattamente in forma di frazione di due interi**

L'ingegnoso metodo di Archimede per approssimare il valore di π

Ma come aveva fatto Archimede a trovare quel valore? Seguiamo il suo ragionamento:



"Disegniamo una circonferenza con il diametro di 1m. Questa circonferenza è speciale perché' la sua lunghezza è proprio il valore di .....; infatti, C = 1 • ..... (m) = ... (m)"

"Adesso, inscriviamo in questa circonferenza un poligono di 96 lati."

"La circonferenza sta molto vicina al contorno di questo poligono, ma è un po' più LUNGA"

"Quindi la lunghezza della circonferenza, che è ....., è MAGGIORE del perimetro del poligono, che si calcola essere circa 3,1410 Quindi π > ....."

"Adesso, circoscriviamo alla nostra circonferenza un poligono ancora di 96 lati"

"La circonferenza sta ancora molto vicina al contorno di questo poligono, ma è un po' più CORTA"

"Quindi la lunghezza della circonferenza, che è ....., è MINORE del perimetro del poligono, che si calcola essere circa 3,1416 Quindi

π <....."

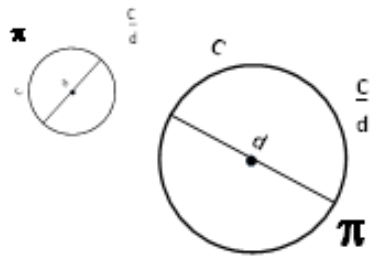
"Quindi il valore di pi-greco è un numero > ..... e < ....."

In realtà, siccome Archimede usava le frazioni, perché i numeri con la virgola non erano ancora stati inventati, lui espresse così il suo risultato: deve essere π > 3 + 10/71 e deve anche essere π < 3 + 1/7

Sintesi: Lunghezza della circonferenza e π

1. Il rapporto $\frac{C}{d}$ è lo stesso per tutte le circonferenze

ed è uguale al numero π, cioè:



$$\frac{C}{d} = \frac{\text{lunghezza della.....}}{\text{diametro}} = \pi$$

π si legge "pi-greco" ed è la lettera p dell'alfabeto .....

2. La particolarità di π

Perché usare la lettera π invece di scrivere il numero?

Perché non possiamo scriverlo come numero decimale! Infatti, se cerchiamo di calcolare tutte le sue cifre decimali scopriamo che non finiscono mai e neanche si ripetono!

Ecco le prime: π = 3,14159 26535 89793 23846 ....

3. Che valore usare per π?

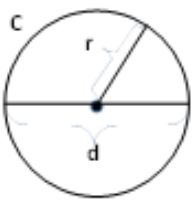
Il valore da usare nei calcoli per π dipende dalla precisione richiesta, per esempio:

usiamo	π ≅ 3,14	per approssimazione al centesimo
usiamo	π ≅ 3,_ _16	per approssimazione al decimillesimo

Si può anche tenere nei calcoli la lettera π e usare le regole del calcolo letterale

4. Lunghezza della circonferenza

C = d ▪	oppure	C = ... ▪ r ▪ π
d = diametro	r = raggio	



Esempio 1: Calcolare C sapendo che d = 2m.
C = ... ▪ π e usando π = 3,14 si ottiene: C = ...▪ 3,14 = 6,28 (m)

Esempio 2: Calcolare d sapendo che C = 25,12 cm.
Siccome deve essere: 25,12 (cm) = d ▪ π allora d = 25,12 : π e usando π = 3,14 si ottiene:
d = 25,12 : ..... = 8 (cm)

-> Svolgi i seguenti esercizi del Base Matematica 2: da p. 129: n. 31, 32 e opzionale n. 35

Applichiamo quanto imparato sulla circonferenza e la sua lunghezza

1. La classe 1A ha preparato una grande pizza perfettamente circolare. La lunghezza della circonferenza è 155 cm.

Devono farla passare da una porta larga 50 cm - senza inclinarla o piegarla. Fai dei calcoli per mostrare se la pizza passa o no dalla porta.

.....

.....

.....

2. Un disco volante ha una circonferenza di 50 m. di lunghezza.

Quanto misura il suo diametro? (arrotonda al centesimo)

....

.....

3. La ruota di una bicicletta è alta circa 1 m. Quale è la lunghezza del battistrada? (arrotonda al decimillesimo)

.....

.....

4. Il primo satellite Meteosat fu lanciato nello spazio il 23 Novembre del 1977 per fornire immagini della Terra per le previsioni meteorologiche.

Nel 1997 ne erano già stati lanciati altri 6.

Tutti questi satelliti girano intorno alla Terra ad una distanza di circa 36'000 km dalla superficie della Terra. La Terra ha un raggio di circa 6'000 km.

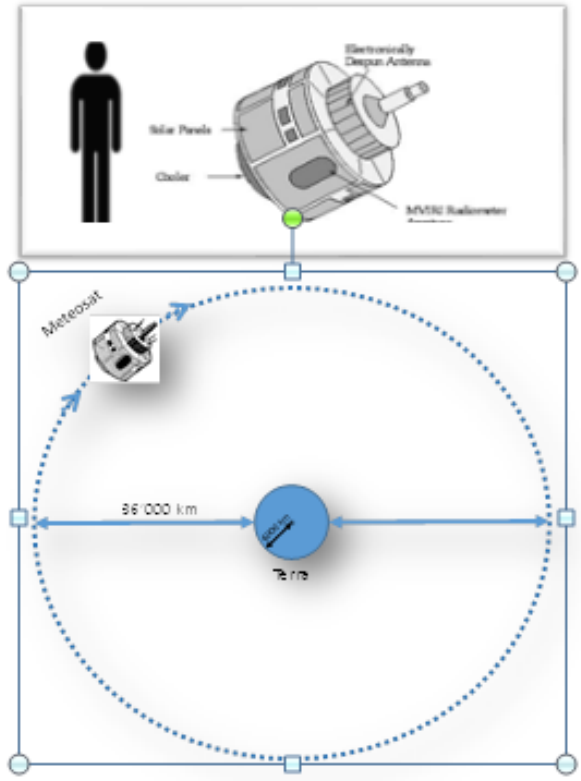
Quanto è lunga la circonferenza percorsa da un satellite Meteosat?

Nota: La circonferenza percorsa dai satelliti si chiama ***orbita.***

.....

.....

.....



5. Quando i satelliti Meteosat diventano troppo vecchi e non funzionano più bene, devono essere spostati su un'altra orbita - per evitare che si scontrino. Quest'orbita è chiamata "orbita cimitero" o "orbita spazzatura".

Lì infatti girano tutti i rottami dei satelliti troppo vecchi.

L'orbita cimitero è 230 km ancora più lontana dalla Terra.

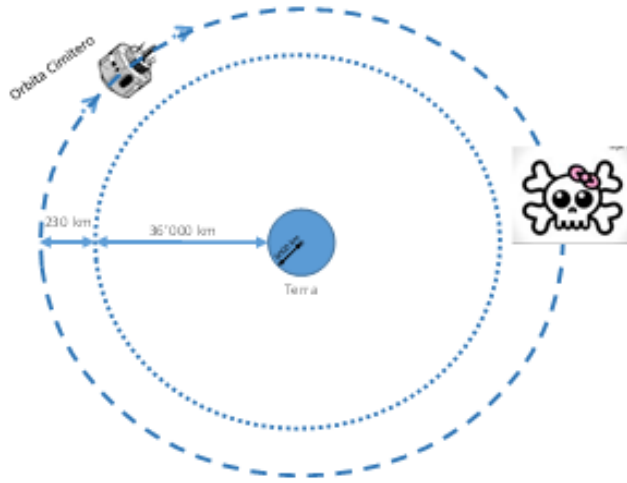
Quanto è lunga l'orbita cimitero?

.....

.....

.....

.....



☺ Rifletti: Se fai 230 ▪ 2 π (km) ottieni ..... che sommato alla lunghezza della prima orbita fa .....che e' proprio la lunghezza dell'orbita .....!

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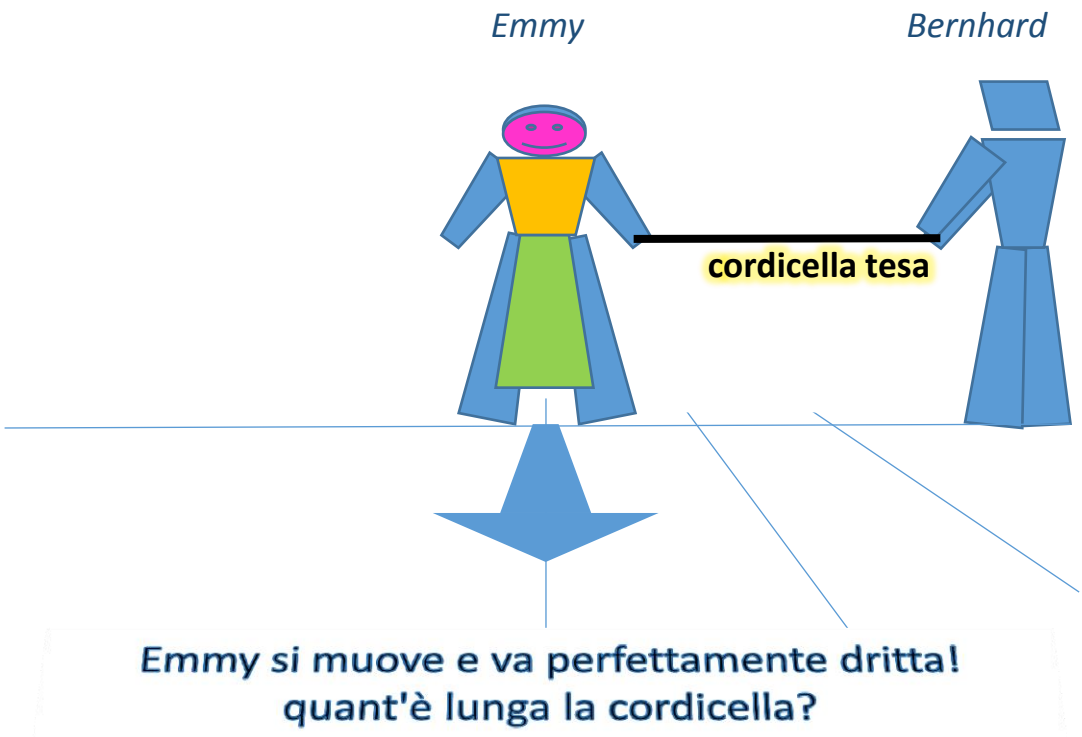
UD 3
Schede per gli allievi

Attività: Circonferenze sulla sfera: Quiz 1

Usate foglio, compasso e sfera di Lénárt per scoprire la soluzione del quiz:



Se non l'avete ricevuta, potete chiedere al docente la scheda guida – se tutti i vostri tentativi di trovar la soluzione non hanno funzionato

Attività: Circonferenze sulla sfera: Quiz 1: Guida

DOMANDE GUIDA

Prendete una cordicella in due: chi si muove va dritto o curva?
……..

Come si chiama la linea lasciata da chi si muove?
……..

Cambierebbe qualcosa se la cordicella fosse lunga 5 km?
……..

Quindi:
su una superficie piatta, una …….. non può essere anche una retta

Utilizzate la sfera di Lénárt:

E se la cordicella fosse lunga 5000 km, cosa cambierebbe?
……..

E se fosse esattamente un quarto del giro della Terra (circa 10'000 km), cosa succederebbe?
……..

Quindi:
su una superficie curva sferica, ci sono particolari …….. che sono anche delle rette

Se aumenti il raggio ancor più, cosa succede alla lunghezza della circonferenza?.....

Riesci a trovare circonferenze più lunghe di queste rette?.....

Quindi:
Le circonferenze che sono anche……. si chiamano ***circonferenze massime perché*** *sono le più ……. che ci sono sulla sfera.*
Le rette *della superficie sferica sono proprio le …….. **massime**.*

Verifica che ogni retta sulla sfera è sempre una circonferenza massima!

Attività: Circonferenze sulla sfera: Soluzione Quiz 1

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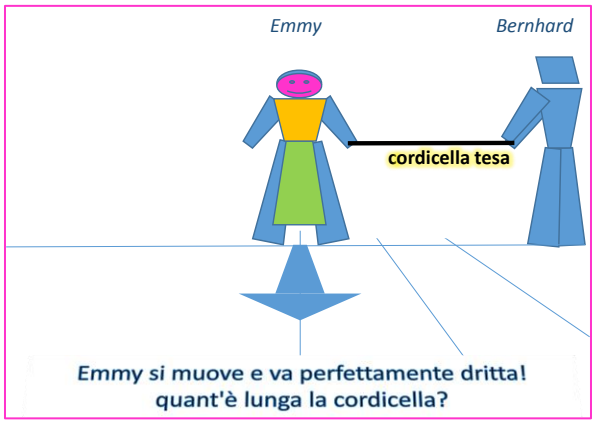
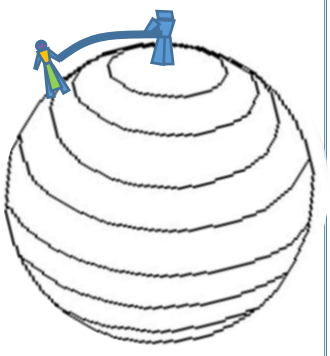
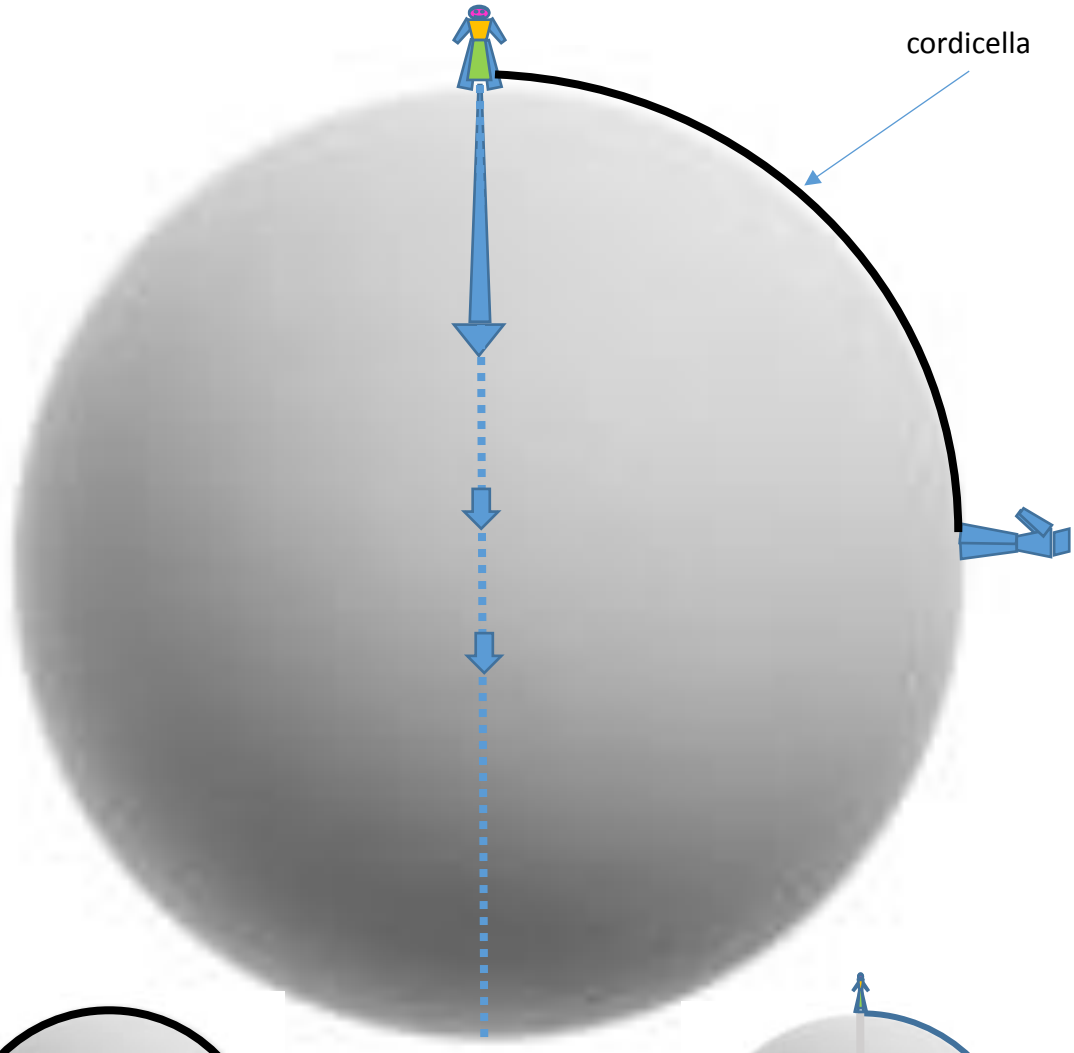
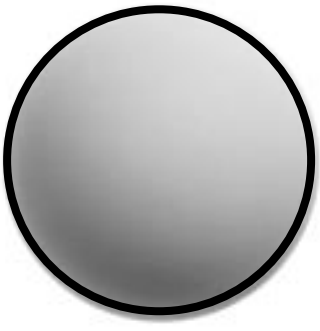
Foglio 3 di 3

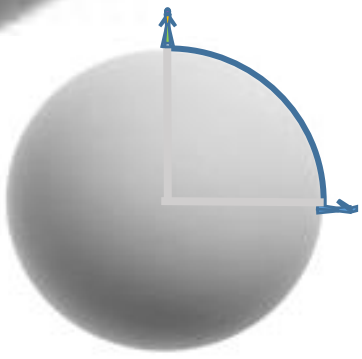








Circonferenza della Terra
≅ 40'000 Km



Cordicella
≅ 10'000 Km

Sintesi: Circonferenze sulla sfera

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Foglio 1 di 2

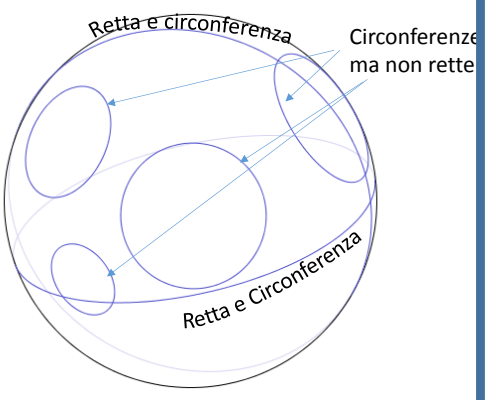
Sapevamo già che **una circonferenza è:**
«Una linea i cui punti sono alla stessa distanza da un punto fisso detto centro, e la distanza si chiama raggio».

Da ricordare

Su una superficie sferica:

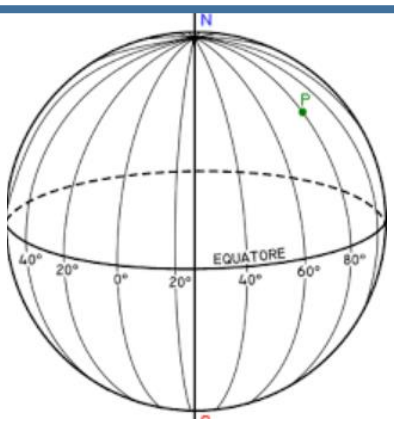
1. le circonferenze possono essere linee curve o rette

2. **le circonferenze ……** sono proprio **le linee rette della sfera**



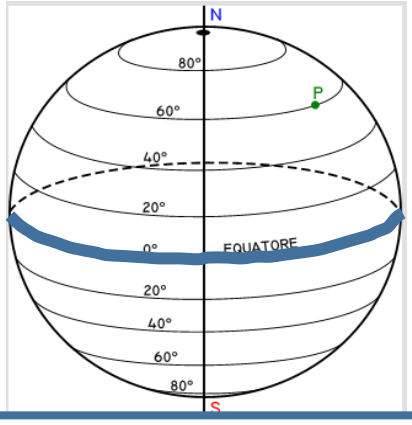
Da sapere

MERIDIANI ed EQUATORE:
sono circonferenze massime e quindi sono …….



Da sapere

PARALLELI:
non sono circonferenze massime e quindi non sono …….

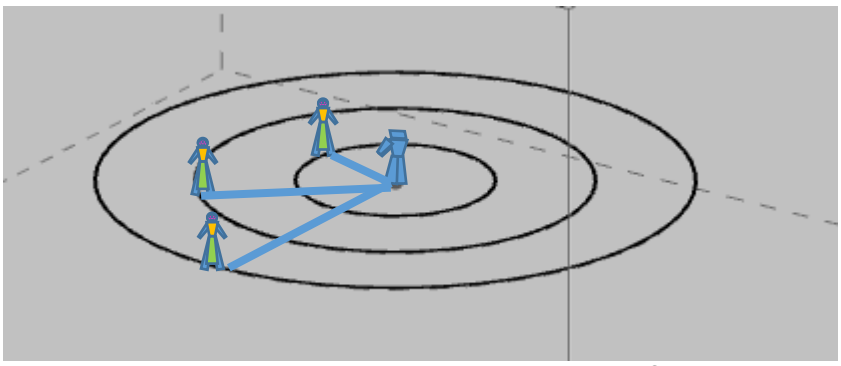


ATTENZIONE:
in geografia l'equatore è considerato un parallelo e quindi è **l'unico parallelo che è una circonferenza …….. e una …….**

Sintesi: Circonferenze sulla sfera

Da ricordare: si sapeva che

Su una superficie piatta (campo da basket o zona di qualche km)

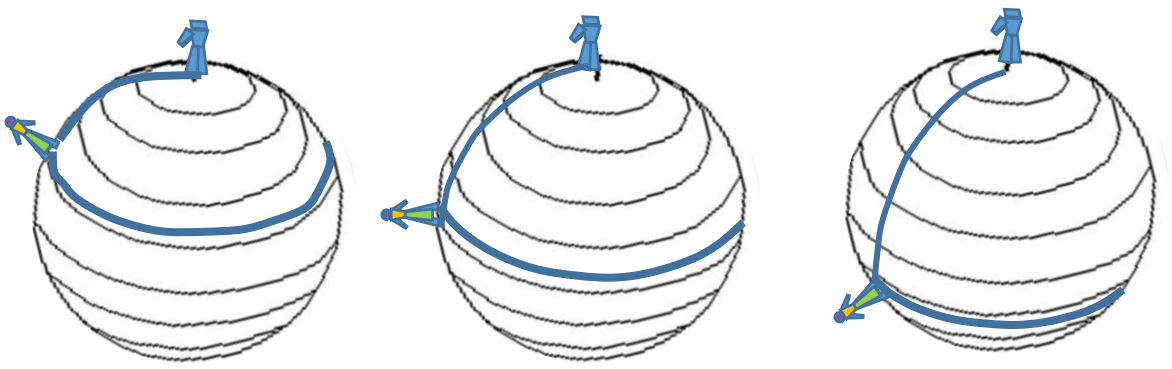


1. se il raggio aumenta, la lunghezza della circonferenza aumenta

2. anzi: il rapporto tra i due è costante e uguale a 2 ….

Da ricordare: abbiamo scoperto che

Su una superficie sferica



Circonferenza = 40’000 km
Raggio = 10’000 km

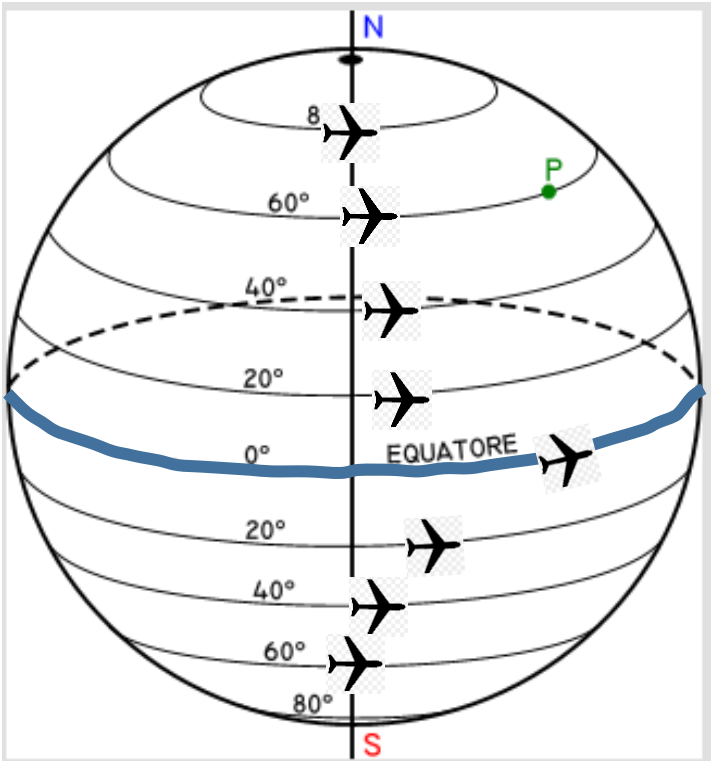
se il raggio aumenta,
1. la lunghezza della circonferenza aumenta,
2. raggiunge una **lunghezza massima** all’equatore e
3. poi ……!

Per l’equatore terrestre il rapporto C/r = ……

Quindi, il rapporto C/r **non è costante**

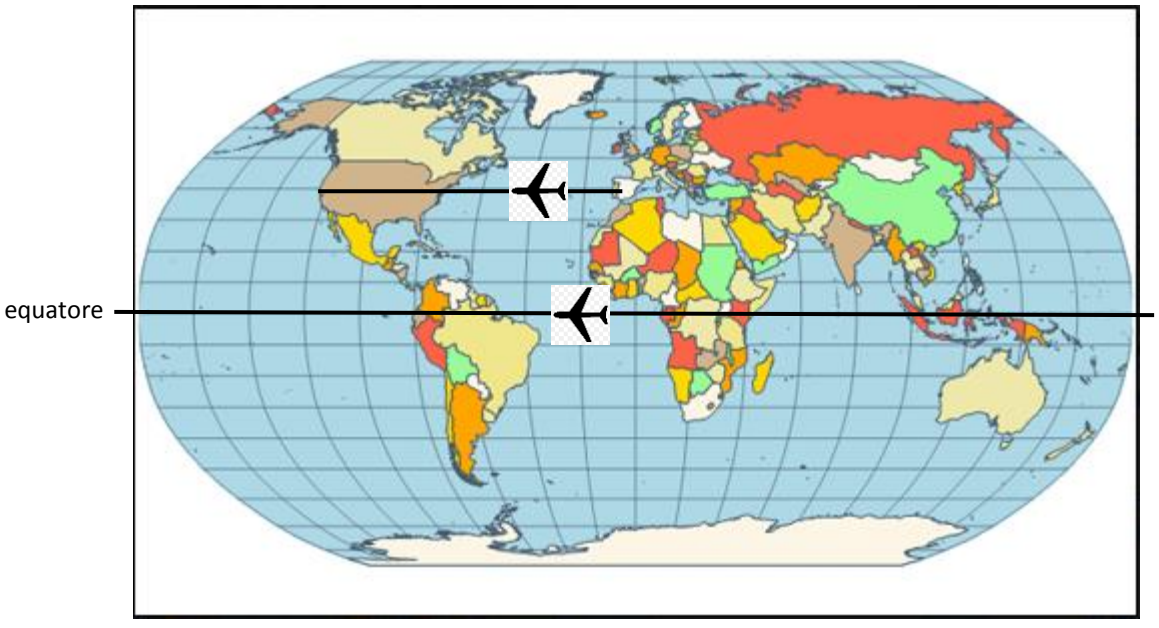
Attività: Circonferenze sulla sfera: Applicazioni

Crocetta gli aerei che vanno dritti (gli aerei seguono le loro circonferenze).



Nella cartina della figura qui sotto:

1. L'aereo che segue l'equatore va dritto? ……..
2. E quello che va da Madrid a San Francisco va dritto? ……..



Attività: Circonferenze sulla sfera: Applicazioni

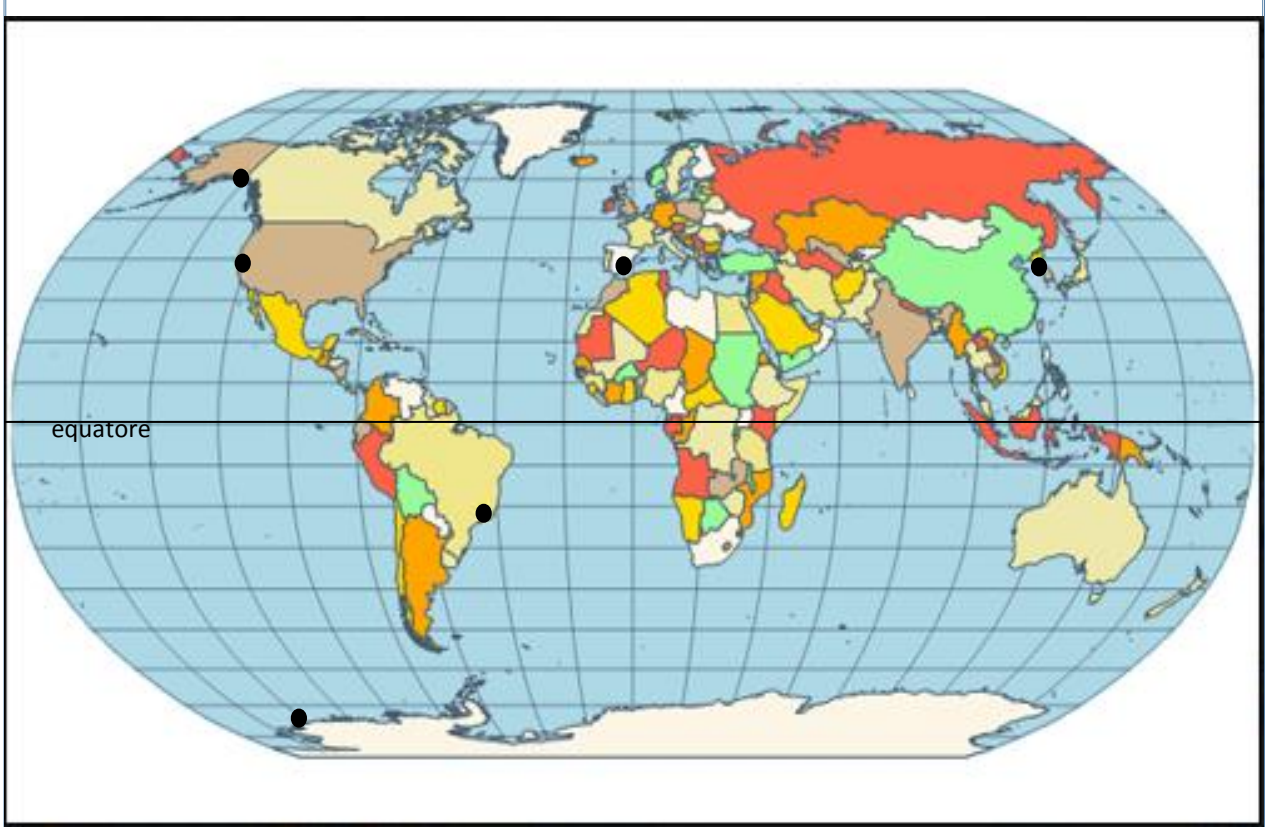
Attività di gruppo

Discuti quali delle rotte tracciate con il nastro sul planisfero appeso alla parete sono curve e quali dritte. Preparati a spiegarlo con l'aiuto della sfera di Lénárt geografica

Quando hai finito, lavora sui punti seguenti:

- Un aereo vola dritto da Seoul (Corea del Sud) a Campinas (Brasile) – indicati da due punti sulla cartina. Scopri sulla sfera di Lénárt geografica la rotta dell'aereo e cerca di disegnarla sulla cartina.

- Ripeti il compito nel caso della rotta dall'Alaska all'Antartide e da Madrid a San Francisco



Osservazioni

1. Una linea retta della sfera sulla cartina si deforma e diventa una curva
2. Se si vola sull'equatore questo non ……
3. Se i due punti sono molto vicini, per esempio, qualche decina di chilometri cosa ti aspetti? …….

Attività: Circonferenze sulla sfera: Approfondimento

Immagina che la Terra sia «veramente» **piatta** e «senza fine e limiti»

Calcola la lunghezza della circonferenza per i diversi valori del raggio e completa la colonna C/d senza fare calcoli (tutte le misure sono in Km)

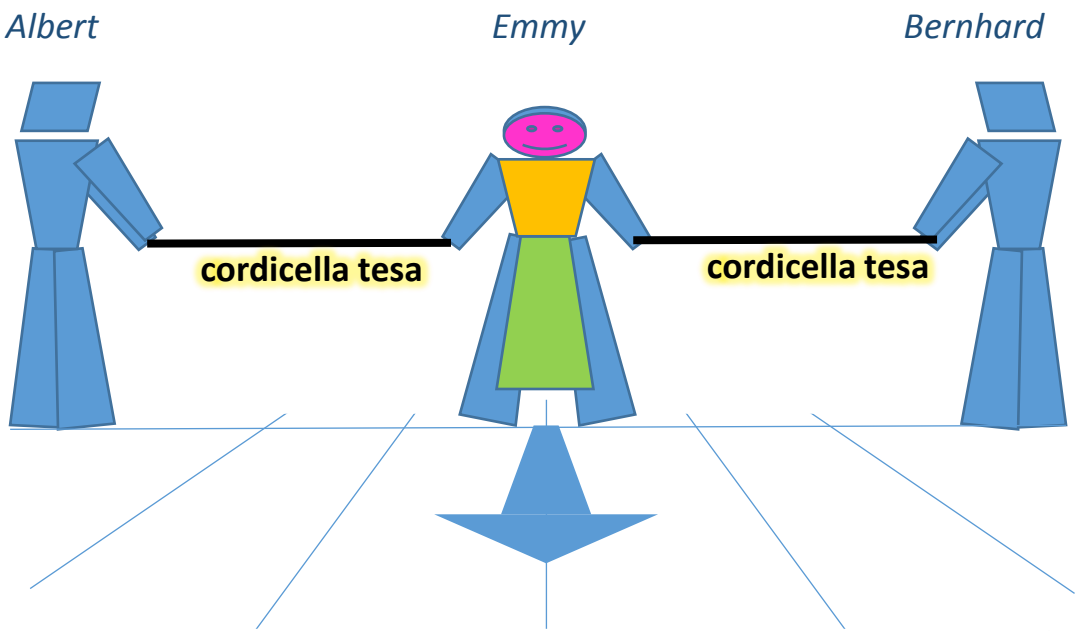
Caso	Raggio	Circonferenza	C/d
1	0		
2	5 000		
3	10 000		
4	15 000		
5	20 000		
6	30 000		
7	40 000		
8	> 40000		

Rifai adesso i calcoli per **la superficie sferica** della Terra (attenzione al rapporto C/d) (tutte le misure sono in Km)

Caso	Raggio	Circonferenza	C/d
1	1	6,28	3,14
2	5 000	28 000	
3	10 000	40 000	
4	15 000		
5	20 000		
6	30 000		
7	40 000		
8	> 40000		

Sulla superficie sferica:
Il rapporto C/d è costante o cambia?.....
E' più grande per valori grandi o piccoli del raggio? .....
Se il raggio è piccolo è come se la circonferenza fosse su una superficie ..... e quindi C/d è uguale a .....
Quale è la massima distanza che può esserci tra Emmy e Albert?
.....

Attività: Circonferenze sulla sfera: Approfondimento



Che lunghezza devono avere le due cordicelle per permettere ad Emmy di camminare dritta?

- Prendete due cordicelle: chi è al centro riesce a muoversi? ……
- Prendete due corde più lunghe: cosa cambia? …….
- Immaginate due corde lunghe 4 km: cambierebbe qualcosa?
  ……..
- E se fossero ancora più lunghe: che cosa succede?
  ……
  ……
  ……
  ……..

Attività: Circonferenze sulla sfera: Approfondimento

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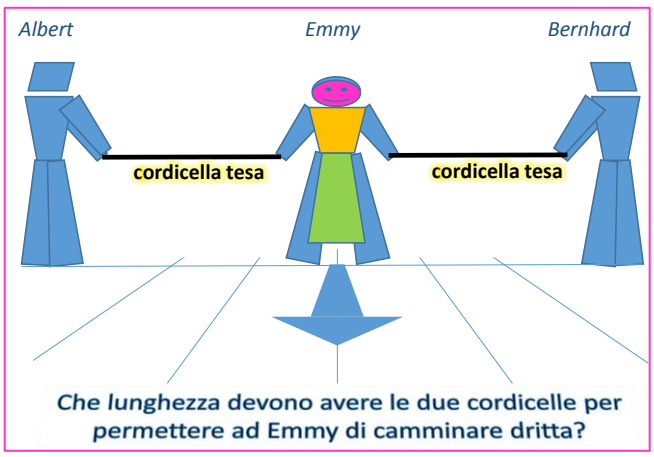
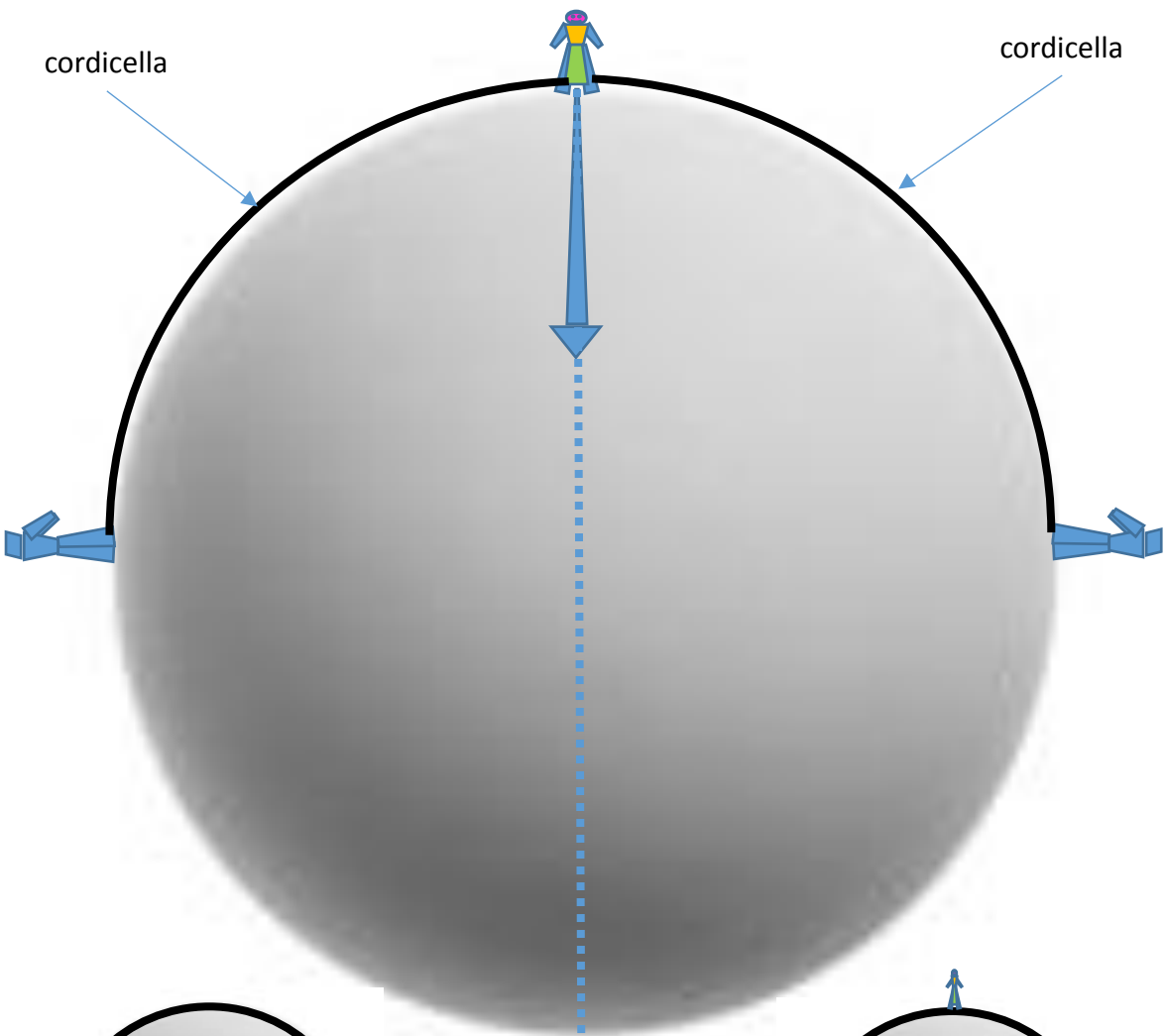
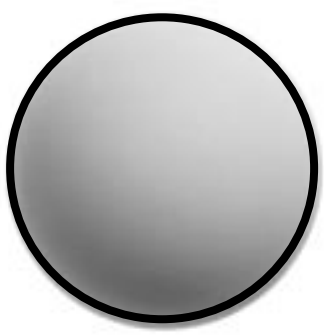
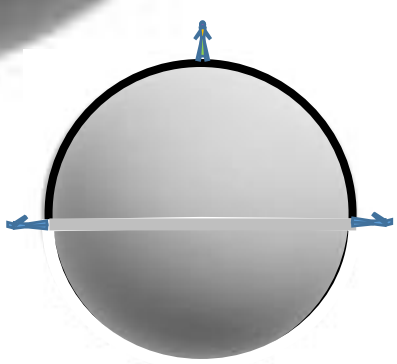
Foglio 3 di 3







Circonferenza della Terra
≅ 40’000 Km



Una cordicella
≅ 10’000 Km

Nota: Se le cordicelle sono più lunghe di 10’000 km, le corde si smollano appena Emmy si muove

Sintesi: Approfondimento: Circonferenze sulla sfera

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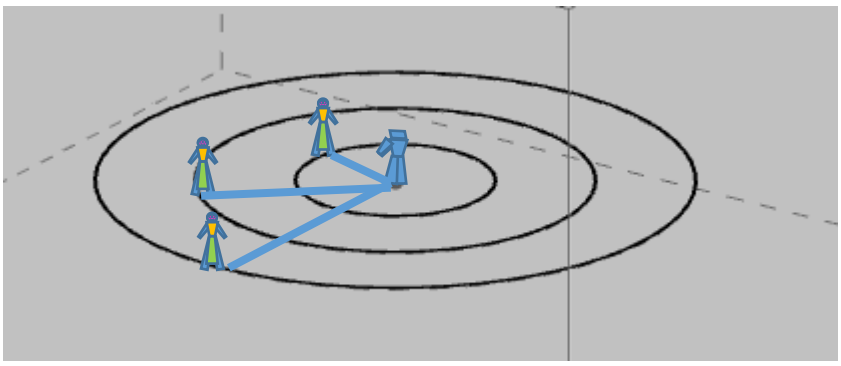
Foglio 1 di 1

Da ricordare: si sapeva che

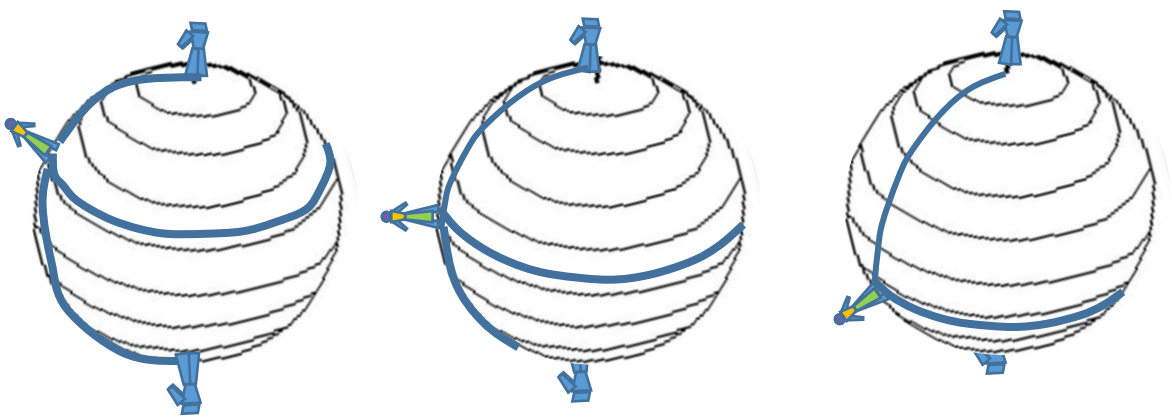
Su una superficie piatta (campo da basket o zona di qualche km)



1. Ogni circonferenza ha un centro e deve curvare per girargli intorno

Da ricordare: abbiamo scoperto che

Su una superficie sferica



1. Una circonferenza ha sempre due centri
2. Giro sempre verso il più vicino
3. Se i due centri sono alla stessa distanza vado dritto

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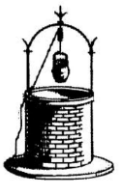
UD 4
Schede per gli allievi

Attività: Triangoli sulla sfera: Quiz 1

Emmy si arrabbia con Albert e decide di fare una lunghissima camminata!



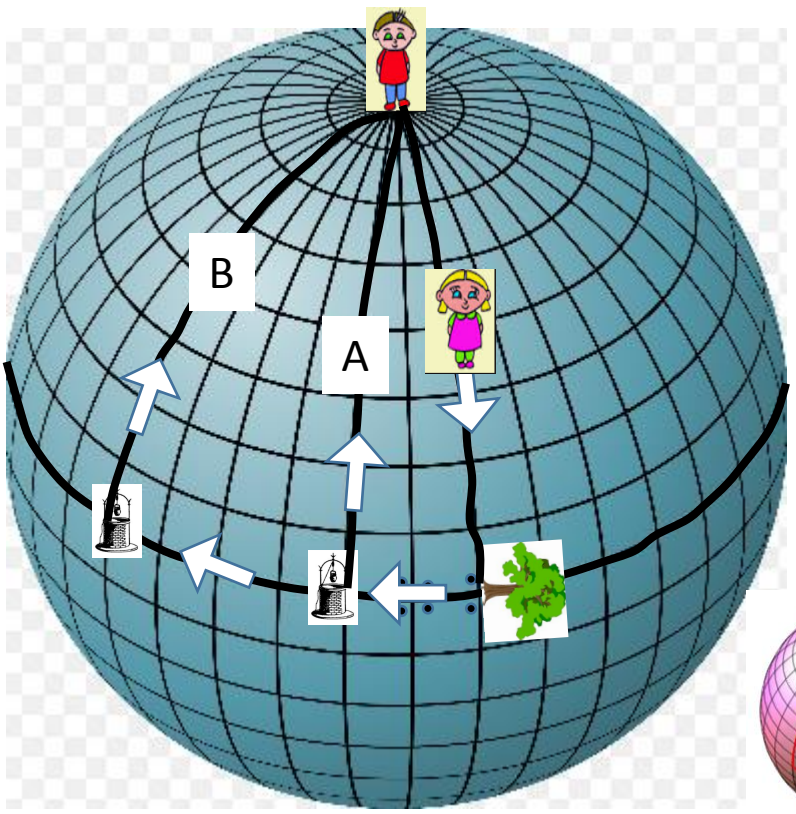
Se ne va via dritta fino ad un grande albero,
dove gira a destra (90°) e va dritta fino ad un pozzo,
dove gira ancora a destra (90°) e va dritta finche' ….
incontra Albert che non si era mosso.

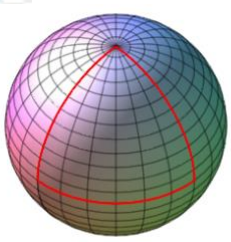
A che distanza è l'albero dal punto di partenza?

Attività: Triangoli sulla sfera: Soluzione Quiz 1

Un possibile ragionamento
Emmy se ne va seguendo una …….. e deve tornare seguendo un'altra …….
Emmy passando dall'albero al pozzo deve seguire una terza …….perpendicolare alle prime due.
I paralleli sono perpendicolari, ma non sono …….. eccetto l'equatore!
Quindi Emmy ha percorso un meridiano, poi si è spostata lungo l'……,
ed è ritornata lungo un ……..

Anche se non sappiamo quanta strada ha fatto lungo l'equatore, possiamo calcolare la distanza dell'albero: è la distanza di un polo dall'equatore: …….





Rispondi:
Emmy può aver percorso diversi triangoli sferici. Nella figura ne mostriamo solo due. Riesci ad immaginarne altri? ……Quanti ce ne sono? …… Riesci ad immaginare quello che ha la somma delle ampiezze degli angoli interni uguale a 5…. ? …..

Misura la somma degli angoli interni di diversi triangoli sulla sfera di Lénárt

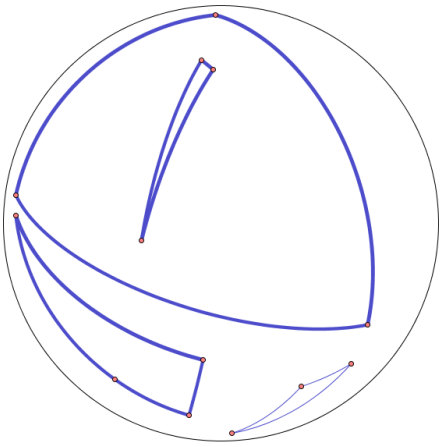
Sintesi: Triangoli sulla sfera

Sapevamo che su una superficie piatta la somma delle ampiezze degli angoli interni di un triangolo è sempre uguale a 180°. Ora invece:

Da ricordare

Su una superficie sferica:
La somma delle ampiezze degli angoli interni di un triangolo è ..... 180°

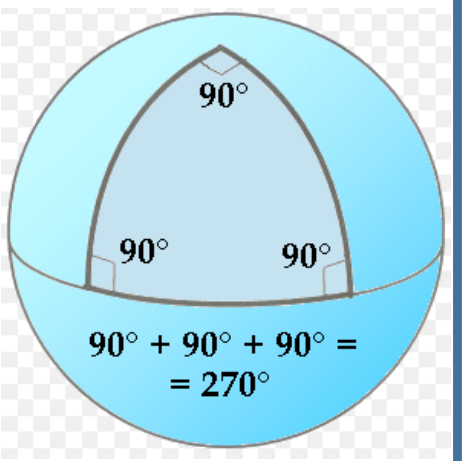
Più il triangolo è esteso sulla sfera, più grande è la somma



Abbiamo incontrato:

Un triangolo equilatero con tutti gli angoli retti (somma = …….°)

Un triangolo con due vertici coincidenti che copre metà della sfera (somma = 540°)



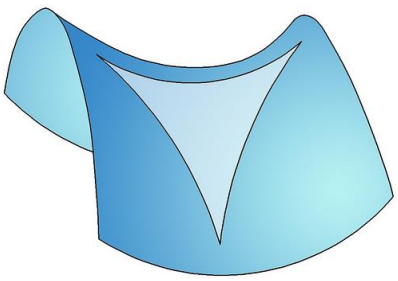
Nota: Se il triangolo è molto piccolo la superficie è approssimativamente piatta e la somma è ……..°

Per Approfondire

Cosa pensi che accada su:

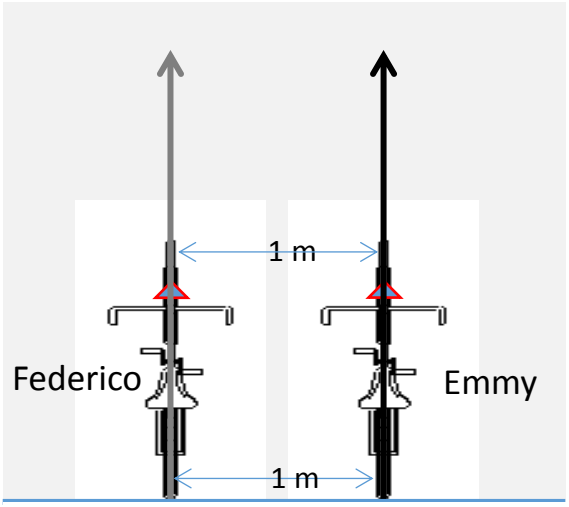
1. un cilindro (puoi prendere il rotolino di carta o una lattina)?

2. la “sella” (il “Pringles”)?



Lavoro di Diploma di M. Banfi
UD 5
Schede per gli allievi

Attività: Parallelismo sulla Sfera: Quiz 1 (sintetico)

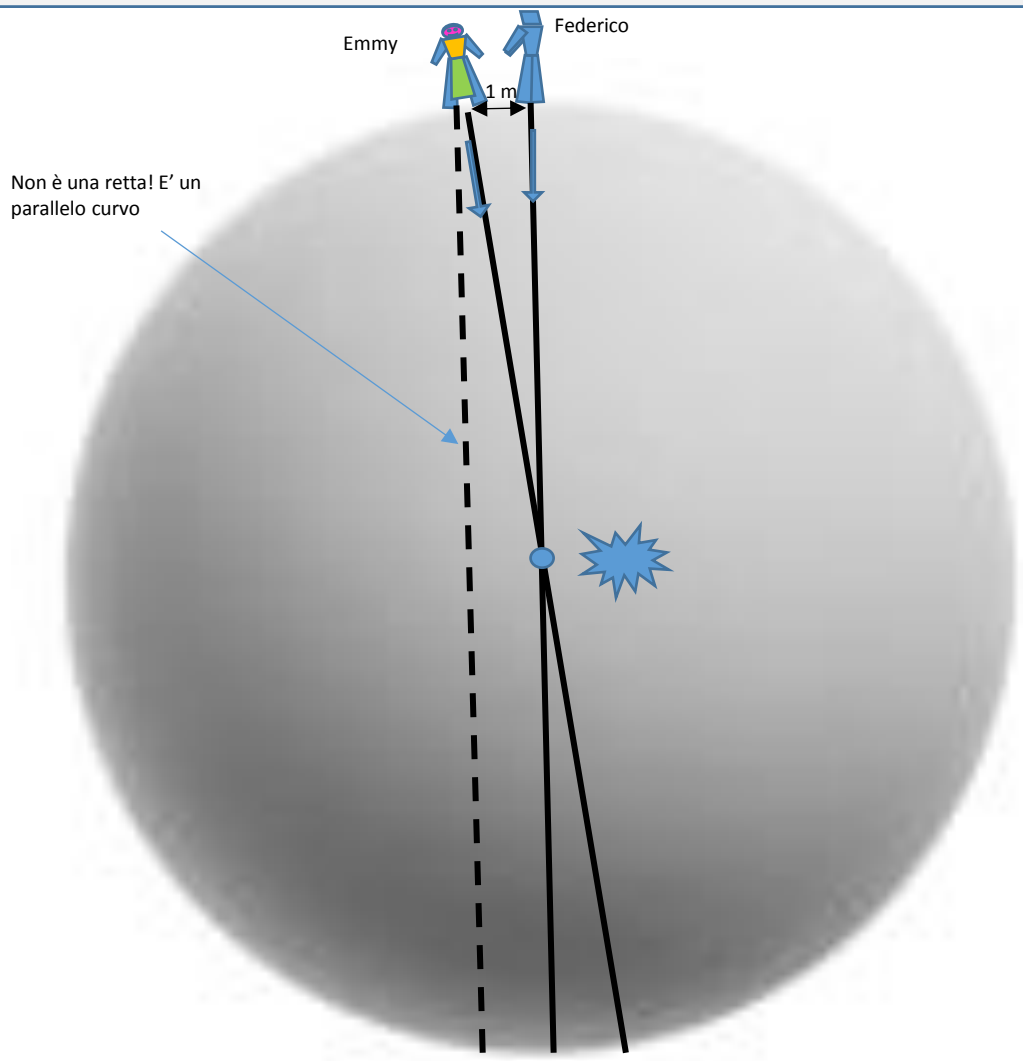


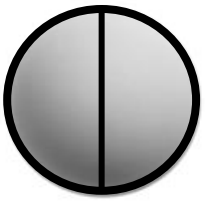
Federico e Emmy salgono sulle bici e si mettono uno di fianco all'altro alla distanza di 1 m - perfettamente paralleli
Vogliono fare un bel giro insieme: il giro della Terra ☺ !
Utilizzate la sfera di Lénárt

Domanda 1 (puoi fare uno schizzo se vuoi)
Se vanno tutt'e due dritti, alla stessa velocità, cosa succede?
.....
.....
.....

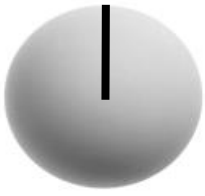
Domanda 2 (puoi fare uno schizzo se vuoi)
Se uno di loro potesse curvare, potrebbero completare il giro?
.....
.....
.....
Se tutt'e due devono curvare, possono completare il giro rimanendo sempre a 1 m di distanza?
.....
.....
.....

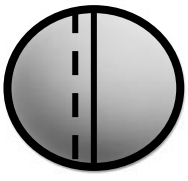
Attività: Parallelismo sulla Sfera: Soluzione Quiz 1 (sintetico)





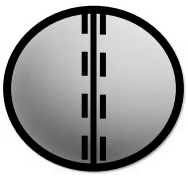
Circonferenza della Terra ≅ 40'000 Km
Un quarto ≅ 10'000 Km



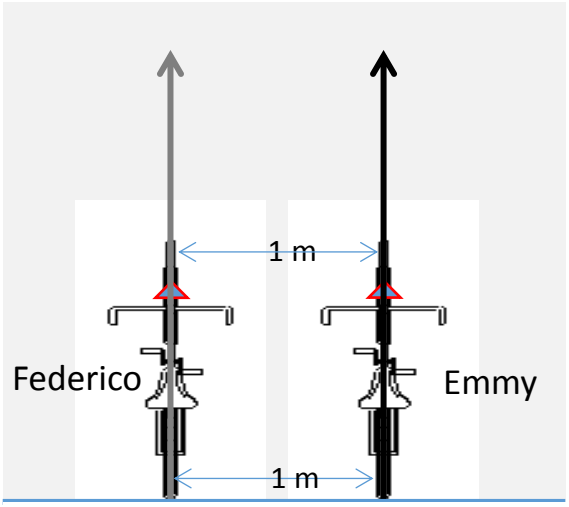


Domanda 1: Si scontrano dopo …….km

Domanda 2:
se uno solo curva (leggermente) ……completare il giro
se tutt'e due curvano (leggermente) ……completare il giro



Attività: Parallelismo sulla sfera: Quiz 1 (Lungo)



Federico e Emmy salgono sulle bici e si mettono uno di fianco all'altro alla distanza di 1 m - perfettamente paralleli.
Vogliono fare un bel giro insieme: il giro della Terra ☺! Ma rimanendo sempre alla distanza di 1 m

La bici di Federico può andare solo dritto. Quella di Emmy può anche curvare (se serve).

Disegnate sulla sfera le tracce del loro giro – facendo finta che la distanza sia non 1 m ma 5'000 km.

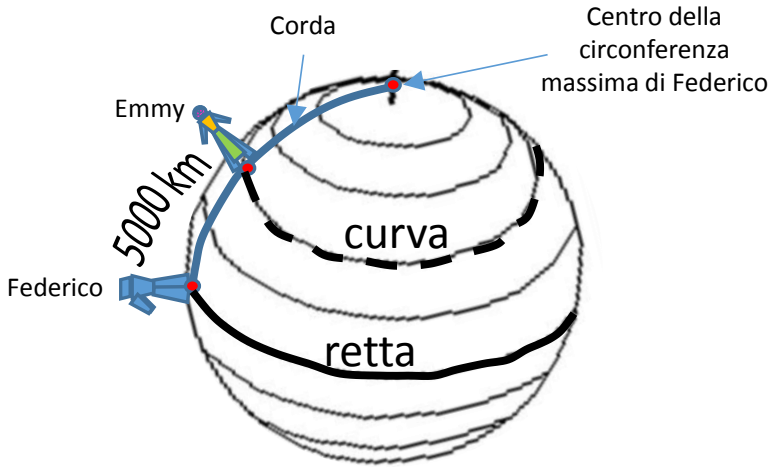
Completate:
la traccia di Federico è una circonferenza ……., cioè una ……. e non curva mai
la traccia di Emmy non è una ……. ……., quindi non è una …… e c……..

Disegnate sulla sfera le tracce del loro giro - facendo adesso finta che la distanza sia di 1000 km

Rispondete
Emmy deve ancora curvare? ……..

E quando la distanza tra i due è di 1 m – Emmy deve curvare o no? ……

Attività: Parallelismo sulla sfera: Soluzione Quiz 1 (Lungo)



Quando Emmy è a 5’000 km di distanza da Federico deve curvare perché è come se lei fosse su un parallelo e Federico sull’equatore

Quando Emmy è a 1’000 km non cambia niente

Anche quando Emmy è a 1 m, matematicamente, non cambia niente: Emmy non è sull’equatore, è su un …… e quindi curva ancora.
(Anche se, siccome è così vicina all’equatore, va quasi dritta)

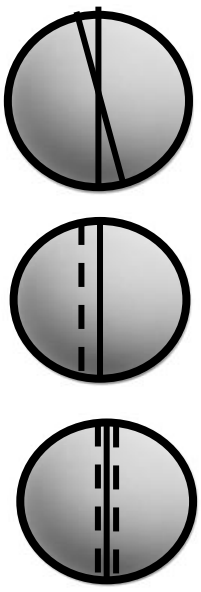
Sintesi: Parallelismo sulla sfera

Sapevamo che su una superficie piatta ci sono linee che sono allo stesso tempo parallele e rette. Ora invece:

Da ricordare

Su una superficie sferica:

Non esistono rette parallele: tutte le rette s'incontrano sempre



Nota: I paralleli non s'incontrano, ma non sono rette (solo l'equatore lo è)

Quindi: Se due linee sembrano parallele

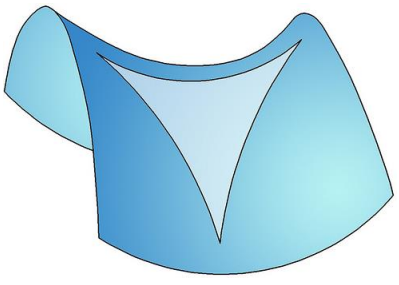
1. o sono tutt'e due curve
2. o una è retta e l'altra è curva

Per Approfondire

Cosa pensi che accada su:

1. un cilindro (puoi prendere il rotolino di carta o una lattina)?

2. la "sella" (il "Pringles")?



Attachment 2: Lectures plans, post-implementation observations and sample transcriptions

Lectures plans, post-implementation observations and sample transcriptions

In the following pages, for each didactical unit (UD), are provided:

1. One-page synthetic lecture plan
2. One-page of key observations or transcriptions of conversations with students

UD	Topic
Entry Assessment	
UD 1	"The Idea" Model of straight line
UD 2	Diagnostic Assessment Earth's Model Riemannian Curvature Straight lines locally Straight lines: non Euclidean behavior
Intermezzo Lecture, Circumference	Circumferences locally Exercises
UD 3	Circumferences: non Euclidean behaviors Straight lines and 2D maps
UD 4	Triangles: non Euclidean behaviors
UD 5	Parallel lines: non Euclidean behaviors
Exit Assessment	

Premise

Four work-lines or streams
Geometry is proposed by multiple work-streams:
- Experiments, static geometry, kinematics, emphatic situations, applications
Please refer to the multiple-streams figure discussed in the chapter of the laboratory design.

Differentiation strategies baseline
When no specific differentiation comments are made, the differentiation strategy is the "baseline" that consists of:
1. Assignments or slides are differentiated only for more advanced students that receive additional assignments or slides for homework.
2. A more advanced student is a student that in that specific UD has a) completed the basic work faster than other students or b) not been so fast, but has shown that he can go beyond the basic work and is very keen on working at home.
3. The teacher puts different emphasis on the principal types of intelligences at work in the laboratory, namely, kinesthetic (or proprioperceptive), visual, logico-linguistic, in discussions and in leading work groups, depending on the (emerging) profiles of the students
4. Groups are recombined as needed to rebalance skills for unforeseen difficulties

A living course
1. The choices of "what to do", "when to do it" and "how to do it" that were made in the planning phase had to be constantly changed to adapt to the response of the students and circumstances. Consequently, the implemented course, presented here, is only a best trade off resulting from an on going adapting and tuning it while doing it.

In hindsight the main change in sequence we wish we could have made is:
1. To start with the work on the model of the Earth – instead of starting with the straight line model. Indeed, the students weaknesses with the Earth model that emerged during the formative assessment, demanded that we took quite a detour into that topic in the middle of the progression from straight line to curvature

The key point that we confirm as vital is:
1. To provide the students with a "actionable model of straight line" before working on any artifact, including the Lénárt sphere - if the students are to experience non Euclidean behaviors as part of their world and not of a toy.

Conventions
1. As in all our thesis, person of female sex or of masculine sex are equally referred to by using the masculine linguistic genre
2. "1 h" means a duration of 45 min (minutes).
3. We used the word course to translate what in Italian would be called "itinerario didattico" and, similarly in French, "itineraire d'apprentissage ou didactique"
4. We use the terms straight line and geodesic interchangeably.
5. We use as synonyms general relativity and geometrodynamics. If not otherwise specified, the text of reference for all of them is "Gravitation" of Misner, Thorne and Wheeler.
6. In geometry, we use the term kinematic to mean the aspects of the movement that we detect with our eyes – in contrast to those that we capture with our proprioperceptive sense (aka, kinesthetic), for which we use the term dynamics. These terms echo the classical (pre-relativistic) physics distinction between Kinematic and Dynamics – in which the "forces" (causes) are part of the second only. When the "orienteering" sense (the ability to frame spatial perceptions to establish relative positions and directions) is involved, we generally use the term "kinesthetic".
7. We found keeping these aspects distinct in the design of the course to be effective for they point at very different perspectives of thinking about motion – kinematic does not presupposes physical (bodily) participation to the movement, differently from dynamics.
8. In psychology of learning, we use the term egocentric as an adaptation of Piaget's egocentric concept, to simply mean the personal, body-centered perspective on the surrounding physical world. Such an egocentric point of observation of the world matches with the didactic transposition of the general relativity fundamental principle of "locality": physics to be simple is to be constructed on the "local" perspective, i.e., Lorentzian space time – and the differentiable manifolds concept of "locality" (i.e., intrinsic local study) – that, applied to our case is: **non Euclidean geometry of the land is built by working locally on its Euclidean geometry**
9. In general, we have preferred to use "un/bounded" instead of "in/finite". The main reason is not to finely mathematically distinguish, but to avoid large epistemological and didactic obstacles. Our students use for the word "infinite" in one sense only: that feature of the collection of the natural numbers for which one can never end counting them. The association of the natural numbers to the points of a design of a straight line is only secondary to them. Therefore, we believe, that using the same word to indicate a feature - not of a discrete set of objects and not related to counting, but of a surface would confuse them and would ground a misconception infinite = unbounded
9. In general, we do not use distances (metric) – as it is common in post-Cartan view of geometry ("without coordinates")

Model of the straight line on the basketball court: UD1

UD1: Launch (UD 1 Slides: p.2) - Duration: 0.2 h
Kickoff: Teacher launches the course by recasting familiar plane geometrical figures (e.g., straight lines, triangles, circles) as theatre characters with, soon to be discovered, surprising and revolutionary behaviors. Teacher selects and names some real and main characters of the history of science and mathematics that, for our purposes, played a key role:
1. in making geometry into a science: Euclid
2. in igniting the non-Euclidean revolution: Gauss, Bolyai and Lobachevski
3. in recasting the Euclidean and non-Euclidean geometries into one powerful structure (manifold), in transforming geometry in kinematics, in re-modelling the space of our direct experience into a four dimensional manifold of entangled space and time and gravitation force as its curvature and in integrating in it energy and matter: Riemann and Einstein

UD 1: Lecture 1 (UD 1 Slides: 3 to 12) - Duration: 2 h
Premise
Traditional Euclidean introductions of straight lines are replaced by this laboratory in which rich and complex personal and sensorimotorial perceptions are used to build a concept of straight movement and line that is portable on any smooth surface. Equipped with this learning, it is the student himself that discovers straight lines on smooth surfaces (sphere) – in lecture 4.
Objectives: To have the student
1. build a complex model of straight line by working in his natural egocentric and local frame to integrate static visual, kinematical and dynamical (body's stress) experiences inter-mediated by language (oral and slides) and pictures (slides)
Aimed outcomes: the student can:
1. talk about what he feels, sees and what trace he leaves (e.g., his ideal bike) while going straight on a small(*) and empirically flat surface (e.g., the basketball field)
2. confirm there is only one straight path from A to B or, equivalently, there is only one straight line passing by A and B
3. talk about the meaning of zero- mono- and bi-dimensionality of points, lines and surfaces.
Experiments and Materials: Several, see the slides.
Organization:
1. groups of three or four turning through different thematic posts (two bike posts; two rolls posts; two analysis posts)
2. groups are homogeneous (i.e., students of comparable cognitive and disciplinary profiles)
3. stronger groups are asked to complete all experiments and report on all of them
4. class discussion is preceded by two separate sub-class discussions: one of the stronger groups, self-managed, and the other of the weaker groups - supported by the teacher
Institutionalization: Two types of slides to be distributed at the end:
1. Results: wrapping up the broad learning facts – to be distributed and then to be used collectively by students to answer the teacher's questions.
2. Synthesis: distilling minimal mathematics facts – differentiated (one complete and the other simplified and called "in pills")
Geometry concepts:
1. Concept of going straight and straight lines on a surface small enough to be in the student visual field and to appear flat
2. First Euclid's postulate: is transposed into an evident truth for an observer of an explicit, specific and materialized environment space: the small flat area
Physical sciences concepts: None relevant
Geometry pedagogy/ cognition:
1. The student cognitively associates straight lines behaviors to an underlying space (he will become aware of its importance in successive classes)
2. The student is supposed to re-organize his preexisting schemes, images, intuitive models and school geometry models in the proposed model of straight line
3. The target cognitive geometrical model of straight movement and line is developed by processing egocentric outdoor personal experiences "attached" to the basketball field
Epistemology:
1. Straight lines of general relativity on a locally Euclidean manifold
Obstacles and strategies:
1. The selective intelligence to abstract from the many and diversified experiential and cognitive data (including any type of outdoor emotional distractions) and to focus only on the very few that are required to organize the straight line model (onto-genetic obstacle the difficulty of which changes depending on the motivation, volition and cognitive profile of the student). The strategy is to lead students with the guiding questions of the slides or by interventions of the teacher or the peers
Differentiation strategies: Baseline + differentiated work organization (see above)
Group dynamics strategies: students with a lot of physical energy were put in charge of preparing and resetting the materials outdoor

(*) i.e., deviations from Euclidean flat geometry are too small to be detected by the observer resolution threshold

Post-Implementation Observations

Model of the straight line on the basketball court: UD1

UD 1: Lecture 1 (UD 1 Slides: 3 to 12) - Duration: 2 h

Socio-affective

1. All students categorized the lecture as "non mathematics"; for example:
"This not mathematics … it is cool!" (Giada)
"But …, so we won't be doing mathematics today, right?" (Francesca)
"We ain't gonna do maths today either, right?" (Paolo in the successive lecture – referring to the present)
2. Students of other classes (more senior) that happened to see the class working outdoor, were equally puzzled and asked to be involved them as well (which was not possible): "We did not do this in second grade … Could we do it with you as well?" (an unnamed student)
3. Students generally appearing inhibited to express their opinions, talked a lot outdoor; yet, they reverted to customary more introverted attitudes as soon as they were back in the classroom
We remark that, outdoors, it was vital to give precise instructions and timing to students to keep them focused instead of start "playing" and "roaming around".

Cognition

1. As foreseen, the major difficulty was to have students abstract from the many aspects to focus only on those that are the model backbones. The teacher role was critical to lead them to note the target-aspects.
2. To grasp the concept of straight line as the result of "a free to go movement ", I asked different students to explain their feelings in imaginary situations: "when you turn right and left with a snowboard; when you are inside a car turning abruptly; during the few seconds before smashing into the wall of the ice ring because of your too high speed". The definitive convincing mental experiment was when I proposed to them: "you are lying close eyes on the back seat of the car: what push's o pull's you experience on your body?".
3. Students found funny the arrow on the bike-steer. Yet, it was not easy for them to mentally represent the direction of movement by an arrow – a representation that it would appear so intuitive to people accustomed with vectors. Indeed, it took 3 lectures to have them grasp the concept and its use in defining the straight line.

Traces of this difficulty:
Samuel: The arrow may turn while we go straight…
Teacher: It turns if you turn …. Try it out
Samuel goes off with the bike …
Teacher: So you see: The arrow should tell me the direction I am about going to. It is there just for this purpose.
Samuel: But I can go straight to the pole, or to the bench … there are many different places where I can head to ….
Teacher: Ok. But can you show me … not where you will end up - but only how you start out … that is, the very first little tiny step you are about making … just the very first 10 centimeters… you will move on with the bike?
Samuel: Here we are … (he goes little ahead)
Teacher: Can you put down on the ground the arrow?
Samuel: done it!
Teacher: What does the arrow show you?
Samuel: …
Teacher: If you turn the arrow by 90° - is it still "walking within" your displacement tracks?
Samuel: No goes another way …
Teacher: What tells you the arrow then – before the bike really goes off?
Samuel: the first piece of route….
So, the difficulty persisted. I found then useful to ask them to imagine (I had not foreseen its use to explain the direction, so I did not have one with me) an arch shooting an arrow and think at how the arrow goes straight and stands for the direction of the movement. So, some of them rephrased as:
"The direction is how we aim ("puntiamo") the arrow before shooting it with the arch" (Fabio)

4. The capacity to process logico-linguistically the concepts was limited to few students and, to them only I asked to have appropriate verbal descriptions (in writing, the level was even lower, the main reason I had to forgo its use), while for the other I only asked for imprecise/ raw descriptions – planning to refine them in due course
5. Not only the ability in communicating in writing was plummeting from the oral, but any "test" was taken in the spirit of a "right/ wrong and good/bad" (the "didactic contract" of Chevallier) – rendering it totally ineffective for assessing their real learnings. Of the initially planned 5 collective tests we kept only the entry and the exit tests and concentrated on TEP's and oral discussions analysis.

Learning abilities

1. Weaker students had difficulties in reflecting, post their work, to tell me what they had learnt. Stronger students were at ease and appeared accustomed to doing it. This might confirm that one advantage of stronger students is their "self-aware active learning".
2. Most importantly, this difficulty made more difficult for weaker students to "give a sense" to what they had done and, therefore, impacted negatively on their learning.

Formative assessment on Earth's model (UD 2)

UD 2: Formative assessment on Earth's model (UD 2 Slides: p. 2) - Duration: 0.5 h

Motivation and objective:
Since the students had not done any world's geography or Earth's geoscience, we had no information about their different levels of understanding of Earth's shape. Neither had we any information about their familiarity with the intuitive geometrical concepts of surface, solid, spherical surface and sphere – for they had had no middle school instruction on the sphere, the circle, and solids other than the cube. We proposed a formative diagnostic assessment to get information necessary to decide where to start from with the Earth's model and surface concept.

Assignment:
The assessment was very informal, social and oral with a duration of 30 minutes. I proposed to all groups the following problem:
"Pietro, from the Elementary School of the White Valley, has no idea about what is beyond the mountains range and whether the world is flat and with non ends or is the shield of a turtle or is a disk. You guys have decided to help him get some good ideas of how things are. You may use any atlas, globe, Google maps or ask to me for other resources to prepare an explanation of what it means that the Earth is round. Be prepared to answer his questions – including the following (written on the blackboard):
1. pictures of the Earth show a disk – not a sphere
2. when we look down from Arosio [a place on a mountain top nearby] we see a flat world
3. the Earth cannot be a sphere, because there are mountains and continents that float on deep oceans
4. the Earth cannot be a sphere, because a sphere is round while our neighbor mole says that all her tunnels are really straight - exactly as those of her cousin who lives inside the Cube Planet

Post-Implementation Observations

Formative assessment on Earth's model (UD 2)

UD 2: Formative assessment on Earth's model (UD 2 Slides: p. 2) - Duration: 0.5 h

1. All the students "knew" that Earth is "round".
2. Almost all were clear about the fact that the geographical globe represents the world where we are, but some of them were not able to locate Switzerland on the globe.
3. Only a few students could not answer why we cannot see that the Earth's surface is curved. The other said "It is too large" or similar.
4. When I asked, "Can you show it to me?" they would not know how to do it. And when I asked "Can you make me feel it?" they could not understand what I was after.
5. Many students were not able to resolve the modelling issue of physical landscape.
6. Many students were not able to link Earth's 2D representations with Earth's 3D spherical model.
7. Almost all were lacking of the distinction between the sphere and its surface - and attributed to the former the character of being curved.
8. Most of them had heard of "force of gravity". But very few of them had heard of gravity as the reason for which "we do not fall off the sphere". Other were confused about falling off. But almost all knew that we do not fall off – whatever the reason.

The successive Lecture was designed keeping in mind these findings. We decided not to take any specific differentiation in addition to the baseline, for we did not have enough information to further distinguish the students.

Model of curvature of a manifold and Earth's surface: UD 2

UD 2: Lecture 2 (UD 2 Slides: 3 to 9) - Duration: 2 h

Premise

Students arrive a the concept of model of the Earth's surface by observing pictures, movies, satellite images and comparing them with artifacts. They discuss flat appearance by processing sensorial experiences of different curved objects and their vanishing curvature, while using maps to become aware of the small scale of their experience of land. They already work on the distinction between curvature of lines and curvature of what is "underlying" to them (i.e.: curvature of figures versus curvature of space)

Objective: To have the student

1. understand the sphere and its surface as physical models of, respectively, the Earth and of its surface
2. build a naïve concept (i.e., initial cognitive model) of a "surface's curvature" by working on artifacts to integrate sensorial and visual data through language (oral and slides) and pictures (slides) mediation
3. include as pivotal aspect of his model of curvature the "vanish-ing-on-a-small-scale" (i.e., mathematically, locally flat)
4. recognize that the concept of curvature (Riemannian) parallels his direct experience and knowledge of the terrestrial surface
5. distinguish the key difference between the "curving" of a line and the "being curved" of a surface: deviation from straightness of a movement the first (observable, controllable), and a feature of the "underlying" the second (not observable, not controllable)
6. the "underlying" is "the place of the geometry of figures" (Euclid's), now plotted by geometry of movements (general relativity) and itself an object of geometry (from Riemann onwards)

Aimed outcomes: The student

1. can represent his place on Earth as a point on the geographical globe and can discuss relative positions on Earth as relative points on the globe
2. understands that 2D pictures are just views (results of projections that deform images) and has an idea of how small is the scale of his neighborhoods
3. can report an example of the difference between the spherical surface and the sphere (the hollow hemispherical artifact); this is essential to prevent "parasite conceptions": a curved surface is already the underlying, the 3D solid is not needed (for examples, as for the pringles)
4. has got a sensorial intuitive model of the curvature of a surface that includes sensorial experiences of the vanishing of the curvature's feeling as the touching area shrinks more and more
5. has started reorganize (by a sensorial imagination) his perception of the land as being flat into images of a beyond-horizon curvature
6. can distinguish "the curving" of a movement from the "being curved" of the underlying surface

Experiments and Materials:

1. A trip to the mountain top was planned to build an image of a large plane, but it was cancelled for logistic reasons
2. Several, see the slides.

Organization:

1. all students work in evenly skilled groups on the basics of the 3D model of the Earth, of the terrestrial surface and of its feature of being both "flat and curved"
2. only some students work in highly skilled couples on the in-depth's (e.g., search info about geoids; explore the curvature of cylinder, egg and pringles; the Greeks and the Earth's shape) – while other students continue exploration of basics artifacts and ideas
3. in collective discussions each group shares their findings and debate others'

Institutionalization:

1. It concerned only key concepts and the same synthesis slides were distributed to all students.
2. No synthesis slides are distributed for the in-depth's (these activities remain open explorations for the advanced students)

Geometry concepts:

1. Concept of surface, curved surface and flat surface – distinguished from 3D objects
2. Faked curvature by bending A4 paper sheet
3. Initial intuitive concept of curvature of a surface vanishing on small scale: "at every place the surface is empirically as flat as we want, as long as we make our observations in a region as small as required" – curvature is not passed onto surface lines
4. For advanced students, other models of Riemannian's manifolds, e.g., cylinder, Pringles

Geoscience:

1. Earth as a planet, difference between real surface and a spherical surface; small scale land flatness and large scale curvature
2. First use of grids of maps

Pedagogy/ Cognition critical elements:

1. help the student to build images of flat and free land by pictures and movies
2. Google Earth zoom out (flying simulation) animation helps building perception of curvature growing out of flat land
3. transparent spherical shell help building perception of movements on spherical surface
4. work on the Swiss Ball helps living the curvature as an underlying of conditioning movements

Epistemology

1. key feature of the curvature of Riemannian manifold (vanishing locally) and charts of a manifold

2. Different geometrical models for different aspects of the Earth (e.g.: orography, political, global, local, precise, flight routes)

Obstacles and strategies:

1. Gravity - Though we excluded gravity, we had to keep in count that students may have difficulties and imagine "things falling off Earth's surface". The planned strategy:

a. to have students build a solid mental image that things always fall from our "head" to our "feet"

b. to have them realize that, "the arrow from head to feet" (a pencil tipping in one point of the earth's globe) turns around the globe following curvature

c. to have them conclude that things fall "top to bottom" if we look at them here and now, but from "surface to the center of the Earth" if we look at them in large scale

Epistemology note. The strategy is not that of "shifting from a parallel to a central conception of gravity", but "from a parallel to both parallel & central conception of gravity" - on a small scale (locally) it is a parallel field; on a larger scale (non locally) it is central. This is where non Euclidean geometry becomes apparent within space time structure (tidal effects and geodesics separation of general relativity)

2. 3D Sphere versus spherical surface - Distinguish the solid (sphere) from the surface (spherical surface). This is a key requirement, for there is nothing non Euclidean in the sphere as a 3D figure. The planned strategy: "empathic identification with the mole and the cat".

3. Smooth versus discontinuous behavior – Some students may imagine that at the horizon line the surface abruptly curves down – to reconcile local flatness with global curvature. This is a key point to grasp the essence of spherical (and any Riemannian) curvature: there is no point from which the land starts curving: however far we go, the surface keeps appearing flat. We discarded the use of tessellations, because of their risk of reinforcing the epistemological obstacle by adding to it a didactic contribution. The planned strategy: the galleon video; the giant man waking up (who would not see his feet – see slides); etc..

Differentiation strategies: baseline

Group dynamics strategies: baseline

Post-Implementation Observations

Model of curvature of a manifold and Earth's surface: UD 2

UD 2: Lecture 2 (UD 2 Slides: 3 to 9) - Duration: 2 h

Socio-affective

1. What was said for the previous lecture, at point 1 and 3 could be repeated. In particular, students kept believing that they were not doing mathematics (quite paradoxically, being the curvature of a manifold central to all modern geometry and mathematical physics) and kept being more extrovert than usual.
2. Students attention and interest level fell exponentially when discussion and questions were too subtle for the analytical power of their language (the ability to express in words analysis) and/ or their cognitive processing of sensorial data (discerning solid from surface). We believe that this was partially due to having not differentiated the groups - they were evenly skilled. Had we grouped the students by even analytical and logico-linguistic skills, things would have probably gone much better for the stronger - whereas for the weaker we could have fashioned a different teacher based supporting strategy.
3. Many students enjoyed imagining to be a mole or a cat and could answer questions without needing to have an "artifact into which to play the character". Some of them told me they imagined themselves to be in a video game. And it worked.

Cognition

1. Quite surprisingly to us, students picked up with no difficulties that curvature of the surface is very different from curvature of the movement line. Students had laid belly down to "feel" what "straight lines feel" when they "make long travels" on Earth's land. Students put just a finger to "feel" what "straight lines feel" when they do short trips on Earth's land.
2. To us, this could indicate that many times it is the didactic that we ourselves teachers underwent that makes us think of something as an obstacle: the student may not have that intellectual and abstract "infrastructure " that may hinder the assimilation of some ideas. One case in point, exemplary, is the supposed cognitive rupture of the property of the angles of a triangle and the whole discussion on Kant. We have proposed a speculative hypothesis in our study report – based on the reflections made on the reactions of the students
3. Only very few students had ever observed a large plane, and a hundred kilometers far horizon, from the top of a mountain, and have difficulty to do it by the help of the pictures and videos only. They made it up by reasoning on artifacts. The initial planned outing to a nearby mountain top (Arosio), from which to build the perception, had to be cancelled for logistic reasons
4. We had Swiss balls, thanks to the kindness of the nearby rehab center, and they proved themselves to be really essential to the work with curvature. I asked students to lay on them belly down – extending their sensorimotorial perceptive data, first with close eyes and after with open eyes.
5. Being it relatively bigger, it made easier for them to pick up the idea of a surface locally flat. Asking them to imagine having Swiss balls more and more large and what would happen to their seeing the curvature led to the "glimpse":
Giada: "but then Earth is as a Swiss ball mega-big, so much that we cannot see it anymore".
6. For the galleon YouTube video, many students could not discount the effect of the waves in hiding the hull and, frankly, we were not convinced ourselves either: it looked as if there had been a point after which it started going "down" – which amounts to create a cognitive obstacle to the grasping of (Riemannian) curvature
7. The distinction solid/ surface was clear for less than half the students:
Teacher (showing a transparent hollow sphere): is a surface or a 3D solid?
Marco: solid
Teacher: why?
Marco: we can touch it?
Teacher: the piece of paper is …
Marco: no, … I don't know
Teacher, showing an A4 sheet: this is …
Marco: a surface
Teacher, rolling it as a cylinder, but not completely: and now …
Marco: is the same thing, a surface
Teacher, closes it as a cylinder: and now …
Marco: the same … a surface
Teacher: a surface, yes
Marco: but the sphere is full of air!
Teacher: and this cylinder it is not …?
Marco: no, it is … I dunno
Teacher: if you take the air off, from that plastic ball … the thing which is left, what is it?
Marco: the sphere
Teacher: fair enough, but a 3D, a solid or what?
Marco: a surface
Teacher: yes, if I take off all what is inside a sphere we are left with its surface – the only thing that counts for our movements … why?
Marco: I do not know …
Teacher: For us, moving on the terrestrial surface, does it matter whether inside the Earth there is gold or rock?
This question sprung out a number of speculations and imaginative ideas …and I left it at there.
6. The point that we believe emerges quite clear is that many students (more than half) have yet to develop capabilities to abstract from the richness of physical reality (perceptions) – in particular, to abstract in the direction of the geometrical or mathematical thinking. As for Piaget studies, major part of our students show to be just around the transition phase from concrete to formal operations.

Learning abilities

1. In this case, all students were at ease in talking of what they had learnt – at their own level of linguistic expression capabilities.
2. A blocking obstacle, springing from a simple difficulty we had not foreseen, is: some students, regardless of having prior knowledge about the order of magnitude of Earth's size, had troubles in making sense of Earth's surface size relative to the size of the objects they are accustomed to. If not managed, this difficulty compromise the ability to see Earth's surface as spherical and see himself on it.
Gabriele, commenting on Earth's shape being a spherical surface: The shape of the Earth is different – in reality…
Teacher: in reality the shape is ….
Gabriele: there are the tops of the mountains that make it "spiky" …
Teacher: can you show that to us by a picture?
Gabriele, draws on the blackboard the picture of a sphere with a spiky mountain as tall as half the radius
Teacher: how long is the full go around the Earth?
Gabriele cannot remember, some help him … 40'000 km
Teacher: how tall is the mount Bar (outside the window)
Gabriele does not know, several buzzing … one suggestion 2000 m
Teacher: can you draw the mount Bar on your picture?
Gabriele: it is small … but I do not know (He cannot do it)
Teacher: how many kilometers is 2000 m?
Gabriele: 2 km
Teacher: and the full go around?
Gabriele: 40'000 km … it is really small like a point
Teacher: the mountains you draw, looks very tall then! How many kilometers would you guess?
Gabriele measures with his hands: about 2000 km
Teacher: and the mount Bar is ….
Gabriele: 2 km
Teacher: do we have mountains 1000 times the monte Bar … 1000 monte Bar one on top of the other … 1000
Student: the highest is the monte Everest at 8848 m, that is 8,848 km
Teacher: so how many monte Bar?
Gabriele: 8 and more
Teacher: and on your drawing how would it look like?
Gabriele: a bigger point …

Straight lines on a flat and bounded surface: UD 2

UD 2 - Lecture 3 (UD 2 Slides: 10 to 12) - Duration: 1 h
Objective: to make the student
1. ponder and getting convinced about some features commonly considered as unquestionably true for any straight movement and line (in the successive lecture the student will discover how these features spectacularly fail on a larger scale of the spherical surface)
2. "bolt these features down" in the perception of the surface where he stands – which is a bounded, small and empirically flat region of the Earth spherical surface (in the end lecture the student will discover that the ideal imaginative extension of this region makes up the Euclid's geometry of the plane)
Aimed outcome:
1. The student is convinced, can argue and can back up with feelings, key facts about straight movements and lines that he recognizes as unquestionably true for as much the A4 sheet as a 5 km terrestrial surface (idealized) field.
Experiments and Materials:
1. None special – on request, previous lectures materials
Organization:
1. initially all students work in couples – evenly skilled
2. later, inter- couples discussion
3. finally, sharing at plenary level
Institutionalization:
1. at the end of the lecture and oral
Geometry concepts: See the slide
Geoscience: See slide
Geometry pedagogy/ cognition:
1. We tried to extract those key facts of Euclidean geometry of straight lines of a bounded region that will be shown to fail in the successive lectures.
2. We used four work-streams: kinematical, static geometry, empathic situations and implications on travelling routes, to present it to students.
3. The empathic situations are key to make students live the questions with their body and were asked to stand up and discuss with each other the questions
Epistemology
1. A few kilometers bounded region of our land is a locally Euclidean part of a Riemannian spherical surface - movements and charts are as for any flat surface
Obstacles and strategies:
The questions potentially touch on a all possible basic obstacles of these first lectures. Yet, being the lecture's target a consolidation of preexisting conceptions, only obstacles critical to this objective will be addressed and this will be done as needed during the work in class.
Differentiation strategies:
1. Students with difficulties in thinking kinematically were flanked by stronger students
2. The same for logico-linguistic difficulties
Group dynamics strategies: baseline
1. to create more vivid perception of and interest in the simulation of the empathic quizzes, we have asked the students known to have an affectional bound to imagine being the characters of the quizzes

Post-Implementation Observations

Straight lines on a flat and bounded surface: UD 2

UD 2 - Lecture 3 (UD 2 Slides: 10 to 12) - Duration: 1 h

Premise
The main difficulty of the test was its, in some parts, "abstractness". Many questions concerned things so obviously true that students would automatically reacting (didactic contract, see D'Amore) by seeking in the wording a possible trick or trap and, if they would not find it, they would stop. An additional cause of difficulty was the wording. To make the questions "portable" onto the successive situations on large spherical surface, I had to rephrase them a bit artificially. For example:
We could not say, "If Elena is upset with Marco and she walks away from him going always straight, her distance from Marco increases", because we did not want to use the distance, so we had to replace "distance" with "… she will be farther and farther from Marco"
Similarly, we note that "the farthest" does not imply to have a metric, but only comparing "separation"
Another difficulty was the "didactic contract": any written test was "blocking emotionally" students – they would get concerned about right and wrong and good and bad. This simple questionnaire, thought administered informally, did not make exception.

Difficulty with "flat Earth"
On the flight routes, many students had difficulties understanding that does not make difference to be a the sea level of at an altitude of 10 km:
Anna: I cannot fly straight from one place to the other because I have to curve
Teacher: you have?
Anna: yes, I have to, otherwise I go in space! I have to curve down (she shows with a hand)
Teacher: I see.
Teacher: when you go straight with your bike do you have to curve down in the ground – with your steering wheel?
Anna: no!
Teacher: and if you had to go from Lugano to Rome all straight on our flattened and asphalted planet's surface, would you have to curve down into the ground – with your steering wheel?
Anna: no
Teacher: no, then …
Anna does not know
Teacher: is the planes instead of flying at 10'000 m were rolling as bikes would have they to curve to go from Lugano to Rome?
Anna: no
Teacher: and if they were just one meter above the ground?
Anna: neither
Teacher: yes.
Anna: it is as if they were very very near the surface!
Teacher: you can imagine them so
But our students are not alone in "struggling" with this issue: I found a video showing planes flying very low and, of course, not curving down. The authors contended that this a proof the "flat land" theory...
https://www.youtube.com/watch?v=unILqRkZGKQ&list=PLu1Mc9o7vAzvGefx5Y9e5Vs3iJwP6xMI8&index=2

Note. Actually, contrary to what commonly believed and reported in many science books as well, there is no difference from being at the sea level, or on a plane flying at an altitude of 10 km or on a satellite orbiting at an altitude of 10'000 km: in all cases the straight movement is around the Earth, without curving down – as it should be clear from the explanations we have provided in our discussion of general relativity. The principle to simplify the issue is the one that we have applied throughout the whole work – learning it from "Gravitation" of Misner, Thorne and Wheeler: think locally: you do not have to curve.

Non Euclidean behaviors of land's straight lines: UD 2

UD 2 – Lecture 4 (UD 2 Slides: 13 to 16) – Duration: 2 h

Premise

In the pivotal first step the student is led through the "cognitive identification" (homomorphism)
- of his body and the flat land where he stands (papalona)
- with his avatar on the tiny stamp-size spherical region (papalina), from where he will draw lines on the sphere (see the picture attached in the following page)

No definitions of straight movements and lines of the spherical surface are given or hinted. We expect that the student generate himself straight movements and lines by transferring onto the spherical surface representing his land the straight movements he built in lecture 1. The actual ambition is that the student discover what land's straight lines are - not what straight lines on the sphere are.

In the second part, we look after the cognitive consolidation of the discoveries. We address the point that "our straight lines" are "curving", by first having the student develop a multi-sensorial (no feelings of bending, no visual deviations) and multi-perspective view (from within the surface the lines appear and feel straight; from outside the surface the lines are curving) on the straight lines and secondly working on the language to use paradoxical logic (they are both)

Objective: To have the student

1. build a "binding" between his "body-centric sensorimotorial frame of reference" on the terrestrial surface and "his avatar" on the Lénárt sphere
2. recognize that the straight paths on the land, if seen from space, would look like the straight paths of the spherical surface
3. recognize, with the help of artifacts, that straight movements and lines on land radically change their (Euclidean) behaviors when observed on a larger scale: these behaviors are pin pointed in a questionnaire on p.15
4. recognize that the dramatic changes in behavior are brought about by the curvature
5. realize that these behaviors may be considered and used as indicators of the curvature of the underlying surface or land

Aimed outcomes: The student

1. can transfer "position and orientation" either way between the sphere and the land
2. using the straight line concept, e.g., "do not bend either side", "keep your direction arrow self-aligned", "follow a bi-adhesive strip freely unrolling" – he can draw on the Lénárt sphere the images of his straight paths of the terrestrial surface
3. can correctly draw straight lines on the spherical surface and make them precise by using the spherical ruler
4. can discover properties of the straight lines of the Earth's surface by studying their images on the Lénárt sphere
4. can make sense of the apparent contradiction, "both flat and curved", by resorting to the memories of the work on the Swiss ball
6. can reinforce cognition by memorizing the associated empathic situations
7. can list some puzzling behaviors

Experiments and Materials:

Lénárt sphere, pencils and ruler; papalina and papalona; bike and bike model; paper rolls and biadhesive; arrows; all previous pictures of flat land and earth from space; Swiss Balls; etc..

Organization:

1. One sphere every two or three students
2. In every couple or group, possibly a person with good kinesthetic intelligence – or have a nearby couple as support
3. Students might go outdoor with their sphere on the basketball field to "bind" the image of the world to the "situation" on the sphere
4. For the questionnaire, recombination of students to ensure sufficient logico-linguistic capability

Institutionalization:

1. Students share their interpretations and findings during the questionnaire
2. Teacher spells out the key facts

Geometry concepts:

1. See questionnaire
2. Straight lines behaviors as an effect and an indicator of the underlying curvature
3. There may be infinite shortest paths between two points
4. There may be straight lines that are not shortest paths

Geometry pedagogy/ cognition:

We have provided the students with open spaces to facilitate their experimentation with their body and orienteering sense. They could use again the bike to build mental image sensorially richer. The Swiss Ball helped them to stabilize the concept of locally flat and, yet, globally curved.

Since the student "sees" the straight lines on the sphere as the traces of his movements on the flat land he re-cognizes them as really straight. Yet, it is impossible to exclude the 3D cognition: the same lines do appear curving as well. We think that this is not issue for the kinesthetic intelligence, but becomes it when the model is processed with language – by which we would label them as curving as well. We have anticipated this issue and in the slides we prepared also a psycho-test (a game) for developing in students a capacity to manage "paradoxical logic" (A and Non A)

Epistemology:

Movements make geometry of space become visible: curvature manifests itself in the appearance of non Euclidean behaviors of straight movements that start from a local Euclidean flat space

The lines are straight – it is the space that is curved (curved space)

If the lines are curving – the space is flat (Euclidean embedding)

Obstacles and strategies:

Since the key point is to identify the place where the student stands with the top of a "large sphere" - the model of which is beside him, the required kinesthetic intelligence could be an obstacle. Our strategy was: further to having different artifacts (e.g., papalina, papalona, bike and bike-model), distribute evenly the students stronger in kinesthetic.

At the end, it is the logico-linguistic processing that become key to consolidate the concepts – and the paradoxical logic there required is an obstacle for our class level. Our strategy was: we have changed the group and distributed evenly the logico-linguistic strong students.

Differentiation strategies:

Baseline + differentiated requests to students depending on their mix of kinesthetic and logico-linguistic intelligences

Group dynamics strategies:

We encouraged those students good with sports (sensorimotorial intelligence), that are generally difficult to get interested in more traditional activities, to take leadership and share their strategies to "understand" the straight lines behaviors on the land and on the sphere

Picture of Papalina and Papalona

Post-Implementation Observations

Non Euclidean behaviors of land's straight lines: UD 2

UD 2 – Lecture 4 (UD 2 Slides: 13 to 16) – Duration: 2 h

The "Toy" risk
The main difficulty was, that as soon as the students were in front of the materials of the Lénárt sphere, they hurried up to take it as a toy and make guesses about straight lines. We had to rewind everything back to the "starting position": avatar on the top of the sphere, papalina and papalona in position – and make them "think", before "doing", about what they would feel and see if they went straight on the ground and how this would translate on the first tiny movement out of the top of the sphere.
Kinesthetic/ Sensorimotorial processing
Yet, even when we managed to take them back to this "initial cognitive settings", the cognitive processing of their body-centric sensorimotorial perceptions was not as sharp as we had expected. Certainly, a good 50% or more of the students immediately picked it up that they had "to stay in the center of the spherical surface" (Euclid's "evenly") and they quickly concluded that a straight line starting from the top has to pass through the "bottom of the sphere" (they meant, antipodal point). However, we noticed that they had a very limited awareness (ability to analytically distinguish and process) of their body centric sensorimotorial perceptions. Some of them would even confuse "front direction with back direction", when they had to associate the land to sphere surface (an orienteering skill).
It is true that they do "live geometry though their body", but, if we may put it in metaphorical terms, quite "literally": their "conscious cognition" is limited. They required to be accompanied, to analyze their own perceptions.
Unfortunately, we did not have time to investigate to which degree there could be a correlation with practicing sports that require a high "kinesthetic awareness" (e.g., classic dance, free climbing) or with practicing outdoor activities that develop frame-based perceptions (e.g., orienteering).
Direction concept
Four students said that any line was a straight line on the sphere. They reasoned by observing that on the ground (on the papalona) we can start off in any direction to go straight and, therefore, on the sphere we can go anywhere. When they saw that a tape had to make a torsion to follow a curving line on the sphere, they correct themselves. Actually, when we asked them to use pens to symbolize the direction on the sphere and show me the self-parallel paths, we understood that the concept of direction was still a barrier. Indeed, on the sphere model, it is required to think of "infinitesimal vectors" (that is – the affine vector does not work because goes off the surface).
Persistence of mathematics reductive-images
Another aspect is that two students pretending to use a ruler (in this phase they had not be allowed to use the spherical ruler) told me: "this is the way we drew straight lines on the plane". When I asked why they would not use what they had learnt about tracing straight lines as traces of their straight movement, they answered that "these are not the straight lines of mathematics". I interpreted it as a symptom of the robustness of the "didactic contract".
I report some parts of dialogues on other issues:

Extract 1: gravity
Francesca: When we move of the top, we fall off the sphere
Teacher: Do we fall off the Earth if we bike to South Africa?
Francesca: ah! Yes, the gravity force

Extract 2: amazing perspective
Teacher: If you are on your bike – and you go straight, which side do you bend?
Marco: No side – I stay in "upright"
Teacher: So, how do go off on the sphere?
Marco: I roll the sphere to leave the bike on top, instead of moving the bike

Extract 3: solid
Emmy: the straight lines are those that come back to the top
Teacher: then this is a straight line (draws a curving line from the top and back)
Emmy: no, because it has turned right
Teacher: ok. Now: if I am in this point on the side of the sphere how can I draw a straight line?
Emmy: I turn the sphere [she means put the point on the top] and pass by the bottom (antipodal) and back

Extract 4: small scale
Samuel: the papalina is "curved"
Teacher: so?
Samuel: is different [from the papalona on the ground that is flat]
Fabio: no, it seems curved, imagine that you are as small as a "bacterium" – your papalina is flat as ours [he showed with his fingers a shrinking of the papalina on the top of the sphere making it so small that would be flat]

Extract 5: Teacher debunked
Liana: Why should we say that the straight lines on the sphere are curving?
Teacher: to me they are curving ..
Liana: but they go straight anyhow …
Teacher: what about this one [draws a straight line on a flat paper sheet]
Liana: it is straight
Teacher: is it not more straight than the other? [on the sphere]
Liana: no the bikes [imaginary] go straight

Liana was one of several students that showed having learnt the concept of "going straight" as related by independent from the "appearance of the line":
1. While the line (static) appears differently (straight or curving) according to the point of view – and creates the issue
2. The act of movement is built in the egocentric perspective: it is perspective invariant – it is always straight

Indeed, light does not bend at all: it goes straight (general relativity)

Intermezzo Lecture: Introduction to the circle and circumference

Intermezzo Lecture:
Introduction to the circle and circumference (Intro Circle MS Word: 1 to 10) – Duration: 2 h
Exercises (Intro Circle MS Word: 10 to 13) – Duration: 1 h

Premise
In this lecture the students learn the answer to: "what is it that makes a circle a circle?" in the traditional Euclidean plane settings, and they build a kinematic or generative model of it. Both the answer and the model are portable "tel quel" on curved surfaces.
Strictly speaking this lecture was not part of the course. Yet, w needed students to have a sensorimotorial concept of the definition of circumference and knowledge of the invariance of "Pi". So we designed the lecture to assure these elements.
Please refer to the slides.
Comments
We wish to comment only on the definition. One key objective was to transfer to the student the idea of a circumference as:
1. the figure that we obtain if we fix a point (the center), cut a rope of fixed length (the radius) and mark on the ground all points that can be connected to the center by the fix rope – either statically (one by one) or kinematically (by a revolving point)
2. Outdoor, with the students in groups, we cut many ropes of the same length and had many students connected by the ropes to a student in the center, so that they could see "how a point of a circumference feels about itself and looks at the other fellow points (static)
3. Then we had one student going around another student (center) and keeping the connecting rope straight, so that they could see how the trace of a circumference is made by movement (kinematic)
4. Critically, we avoided to associate to the circular motion
A. the transversal stress/ force (rope tension – dynamical perspective)
B. curving character
The reason is that such tension existed there, but, that tension, with all transversal stress/ forces vanishes when the circumference radius grows up to (circa) 20'000 km – because the circumference becomes a straight line! If we had included tension in the definition – we could have not arrived at the great circles as the limit case of a circumference when it becomes a straight line. The same applies to the curving character: it is not a feature of the circumference (it is the curvature of the underlying that makes the great circles close)
Of course, these concerns had sense for our static/ kinematical/ dynamical approach to geometry – not for traditional geometry approaches.
In conclusion: our work with the circumference had be "portable on non Euclidean settings on the sphere" to allow students to discover:
1. Parallels are circumferences
2. Equators are circumferences that are straight (fix distance paths, they do not curve, they do not stretch the rope)
3. Length of the circumference increases with the length of the radius-rope slower and slower and when the radius rope goes above 20'000 km decreases

Post-Implementation Observations

Intermezzo Lecture: Introduction to the circle and circumference

Intermezzo Lecture:
Introduction to the circle and circumference (Intro Circle MS Word: 1 to 10) – Duration: 2 h
Exercises (Intro Circle MS Word: 10 to 13) – Duration: 1 h

As post-implementation observations we wish simply report the following

1. Our students had difficulties with the concept of ratio – at their level they had knowledge only of division of integers and fraction as an operator. The concept of ratio of two numbers was still unstable, let aside the non rational character of it
2. The kinematic work outdoor was very effective to pass the idea of the circumference as the trace left by a point that moves at constant distance (separation) from one fix point

Non Euclidean behaviors of circumferences: UD 3

UD 3– Lecture 5 (UD 3 Slides: 2 to 6 for all students – from 9 to 12 for the strongest) – Duration: 2 h

Premise

There are two set of activities: basic and advanced (approfondimento). Basic activities concentrate on qualitative aspects. Advanced require students to fill in a table of the "pi" values decreasing from π to 0 and passing by 2 in correspondence of the longest circumference, and to refine their understanding of the circumference. Cartographic study is the topic of the successive lecture.

Initially all students work in couples: they are connected by a rope and one goes around the other. They observe that "the longer the rope, the longer the circumference".

At this point, students are taken into the riddle (p. 2). They are asked to use the cognitive identification of positon on land with their position on the sphere, and draw, with a string and a pencil, or with the spherical compass, some circumferences on the sphere.

They discover that:

1. the circumferences of their land, and of the sphere, with a radius half way from one pole to the other (i.e., 10'000 km for the Earth model), are straight lines – they do not curve anymore
2. since their length decreases, if the radius is further increased, "the circumferences that are straight lines are the longest" (and the students are told that they are called great circles). Every great circle of their land is a straight line of their land
3. the ratio between the circumference and the diameter is not constant

Advanced students proceed further as explained above.

Objective: To have students

1. discover that the unquestionable quasi-logic truth that "the circumference is curving" fails on a larger scale of his land (and on the spherical surface)
2. recognize that, vice versa, if a line (of his land) is straight then it is always a circumference – not in the sense that is curving (!) But that it keeps itself at the same distance from a fixed point … (!)
3. recognize that among all possible circumferences (of his land) the one that is straight is also the longest
4. consolidate learnings in: a great circle is a closed straight line and a closed straight line is a great circle
5. discover that the ratio C/d is not constant: it is π for small regions and it is 2 for the straight circumference
6. recognize that the ratio C/d goes to zero
7. "Parallels" are not straight lines, because they are circumferences, curving, with the exception of the equator: a straight circumference
8. Meridians are all closed straight lines connecting two antipodal points

Aimed outcomes: The student

1. can use the flat surface defining criteria for a circumference to draw it on the spherical surface
2. can show why a great circle is both a straight line and a circumference
3. can show qualitatively that the ratio C/d is not constant and that C and d grow together only up to the great circle – beyond, C decreases while d increases
4. can use the Lénárt spherical compass
5. may recall paradoxical, empathic situations
6. understand parallels as curving (roped) movements – with the exception of the equator

Experiments and Materials

Outdoor flat ground, ropes, chalks, Lénárt sphere, strings and pencils, spherical compass, etc.

Organization

1. One sphere every two or three students
2. In every couple or group, a person with good "manuality" (to use compass and strings)
3. For the collective sharing, return outdoor to fix the image that it is the curving circumferences of the ground that will become straight – just before curving in the opposite direction

Geometry concepts

1. Circumferences on a spherical surface, parallels and meridians

Geometry pedagogy/ cognition

1. Ensure that the images built on the sphere are transferred into the student world by constantly alternating work on the sphere and work outdoor
2. Avoid linguistic traps, by using a physical and kinematic model of the circumference
3. Attach empathic contents by making the real students characters of the "scenes"

Epistemology

1. Key character of spherical non Euclidean geometry: circumference length goes to zero

Obstacles and strategies

The main obstacles are didactic. The class had only the Intermezzo Lecture about the circumference – learning was still unstable. The strategy: to start from the ropes outdoor and recall circumference key ideas

The π obstacle. Although we had taken care in the Intermezzo Lecture to distinguish between the ratio and π, several students already have a misconception of their identity. The strategy: we manage them on an ad-hoc basis

The parallels obstacle. If they are misconceived as straight lines, the strategy is that of simply getting back to their kinematic origin: curving circumferences (rope and pencil).

The understanding of the "ratio variability" can be a didactic/ epistemological obstacle – for it may require a «function thinking» (as it is a tabular plotting of it). The strategy: differentiation – requested only for advanced students

Finally, the obstacle of processing the logico-linguistic expression "a circumference that goes straight". The strategy is to have the student use the rope on the sphere to see with his eyes what it means

Institutionalization

Key reflection points for all are in p.3 (all students should have developed a valid opinion about each one of them)

Key findings for all are in p.5

Additional discoveries of general interest are institutionalized as required

Differentiation strategies

More advanced students receive only the riddle (p.2)

Less advanced receive also an "investigation trace" to help them crack (p.3)

Advanced students proceed to advanced contents of slides (p. 9 to p. 12)

Post-Implementation Observations

Non Euclidean behaviors of circumferences: UD 3

UD 3– Lecture 5 (UD 3 Slides: 2 to 6 for all students – from 9 to 12 for the strongest) – Duration: 2 h

Less than half of the students managed to resolve the quiz. The other, even with the step by step slide, would not see the solution.
Talking with them, we understood that the difficulty was the number of factors at play:

1. Thinking in terms of ropes extending behind the horizon
2. Remembering the length of a rope from a pole to the equator
3. Seeing everything represented on the sphere
4. Mimicking the movements on the sphere
5. Processing the quiz language

Indeed, when the students understood the solution only very few of them had difficulty to accept the fact that the circumference was straight.
We have analyzed the point with two bilateral and listening to different students .
First of all, many students, still, would not use correctly the word circumference and circle. They do understand that one is the boundary and the other the enclosed region (with or without boundary is irrelevant). When, we insisted that they "should" be surprised, they answered back that "it was evident", "everybody sees that the circumference becomes a straight line on the sphere". They would not be "puzzled" by the verbal description of the situation, "a circumference not curving, but going straight".
We have interpreted this response as already done in other situations, but being at Lecture 5, we were able to be more precise:
We observed that:

1. The student appears to be aware of the fact that, for us, is paradoxical and puzzling and that, we, in the linguistic verbal register, express as "a circumference that is straight"
2. The student processes the fact primary by visual intelligence and secondarily by sensorimotorial – because when he explains it he points to the shapes and mimic them kinematically.
3. The linguistic expression is "tertiary" to him – he is not even receptive to the distinction circumference/ circle.

We make the hypothesis that:

1. The paradoxical or puzzling aspect stems from the linguistic verbal expressions – not by the situation processed by kinesthetic or visual intelligence: we still see a "circle" anyhow!! (see also note 1, more below)
2. The paradoxical or puzzling aspect is amplified by the abstract thinking: the more we divorce the words from the picture or the experience that referred to, and we make them abstract concepts, the more the paradoxical aspect is strong
3. The paradoxical or puzzling aspect is amplified if the person has been used to thinking or seeing or being told "the standard behavior" for years

We draw the conclusion that, if the hypothesis is true, then:

1. The student does not feel anything puzzling or paradoxical because, at his stage of development, the linguistic cognition and intelligence is subordinate to the visual and the kinesthetical (serves to express what he sees – not to discover anything new by "playing with words – for example, in the Russell's tradition)
2. For the student there is not even amplification, because his abstract cognition, if it is developing, it is anyhow subordinate to the other. And, what he has been knowing of the standard behavior of a circumference, he has known it, for a few weeks or, maximum one year

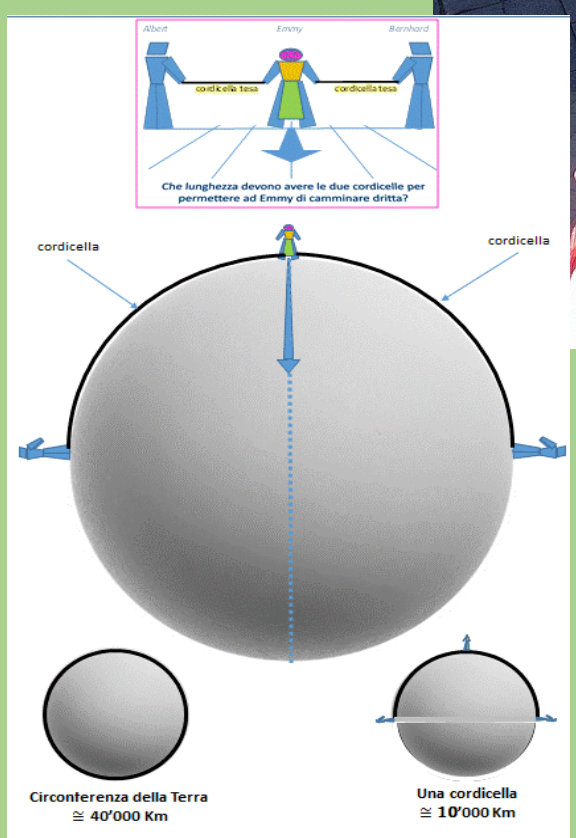
We have found this "schema" applicable to read many reactions we observed in several supposedly-paradoxical situations.

Note 1
We have wondered what would be a really paradoxical fact for a student. We think that this would have to be paradoxical for his sensorimotorial perceptions – not for "notions" that for him are just a code of real behaviors. This confirms that for our student traditional non Euclidean geometries would be without meaning, if presented formally, and irrelevant, if presented on artifacts

Post-Implementation Observations

Non Euclidean behaviors of circumferences: UD 3 ….

The straight circumferences: the only way to be free



Straight paths on 2D maps: UD 3

UD 3– Lecture 6 (UD 3 Slides: 7 to 8 for all students) – Duration: 1 h

A note

The concept of this lecture, together with that of Unit 2, curvature, and Unit 1, straight line, makes possible to understand what we mean in general relativity when we say that light goes on straight path and yet the eclipse of 1913 confirmed the bending of light as predicted by general relativity (see Box, General relativity: revolution of geometry).

Premise

P. 7 introduces students to the problematic – parallels deceivingly look straight in 2D maps. Then, on p. 8 the main work starts. On the walls, two large world maps were prepared, with different cities marked by pins and connected by stretched straight colored strings (see picture in the next page): students have to answer which ones are straight. Then, they can use the Lénárt sphere to check their answers and to discover what is the real shape of the strings (on the world surface). Eventually, they have to pinpoint back to the world maps the real straight paths and observe their distortions. Finally, they complete the questions on p. 7 – reverting to the world map or Lénárt sphere if they have difficulties.

Objective: To have the student

1. discover that the straight lines of his land (and of the spherical surface) are generally distorted by projections on 2D maps
2. discover that the lines that appear straight on a 2D map, generally, are not straight lines of his land (and of the spherical surface)
3. in particular, recognize that, the meridians are distorted into curving lines and the parallels into straight lines
4. discover that straight paths between two cities are projected in a variety of ways depending on the two cities (straight line, curving lines of different shape)
5. recognize that, if the scale is small, then 2D straight lines correspond to land (and spherical surface) straight lines

Aimed outcomes: The student

1. can use the Lénárt sphere to demonstrate and explain distortions between spherical surface and 2D maps (parallels, meridians, straight paths)
2. can use the Lénárt sphere to find the straight path between two cities and pinpoint it on a 2D map
3. is aware that given any two cities the straight path between them may take very unexpected shapes

Experiments and Materials:

Geographical Lénárt sphere; political globes; world 2d map (on the wall); pins; colored strings; etc..

Organization:

There are 2 work-posts, each with a large wall 2D world map and two Lénárt geographical spheres for a group of 6 students (to a total of 12 students). The world maps on the wall are prepared as explained.

There are 6 work-posts, each with an Atlas and a Lénárt sphere (possibly geographical)

Half the students start with wall maps, whereas the other work on questions using the Atlas – exercising the skills of finding and determining straight lines on the sphere (see Differentiation strategy point 2). Later, students switch posts.

Institutionalization:

The key sharing and institutionalization is done collectively on one world map on the wall

That world map , with the colored routes, is left on the wall for the student to revisit it autonomously in the following weeks

Geometry concepts:

2D mapping distortion as indication of non Euclidean behaviors, charts of manifolds

Geography concepts:

Parallels and meridians, sea and air routes distortion on 2D maps

Geometry pedagogy/ cognition:

Have the student work in parallel on the map and the sphere to build images of how straight lines on the sphere are piecewisely distorted into different lines on the 2D maps

Pick up cities they are curious about and build on spot empathic stories (e.g., I want to go to see lions, but grandma and I are in “Terra del Fuoco”: what is the straight flight route? – students find the departure and arrival airport and then study routes)

Epistemology:

1. Straight lines of a curved space are straight – it is their projections that may be curving (light does not ben around the Sun – it is its projection in space that is distorted and “bends”)
2. On small scale a Riemannian manifold is flat, therefore a 2D map preserves lines geometry (locally Euclidean manifold)

Obstacles and strategies:

There is a kinesthetic obstacle if one tries to map a line from the Lénárt sphere into a line of a 2D map, or vice versa, synthetically. The strategy is simple: we lead students to pin intermediate points on the “starting line”, find their location on the target surface and interpolate them there.

There is an initial difficulty in accepting that curving lines on the 2D map are images of straight lines of the land and the sphere (and vice versa). The strategy: to work on several examples with the Lénárt sphere to change perception. We do not anticipate it to be a common obstacle – because very few student seemed to be accustomed to work on Atlas and World Maps.

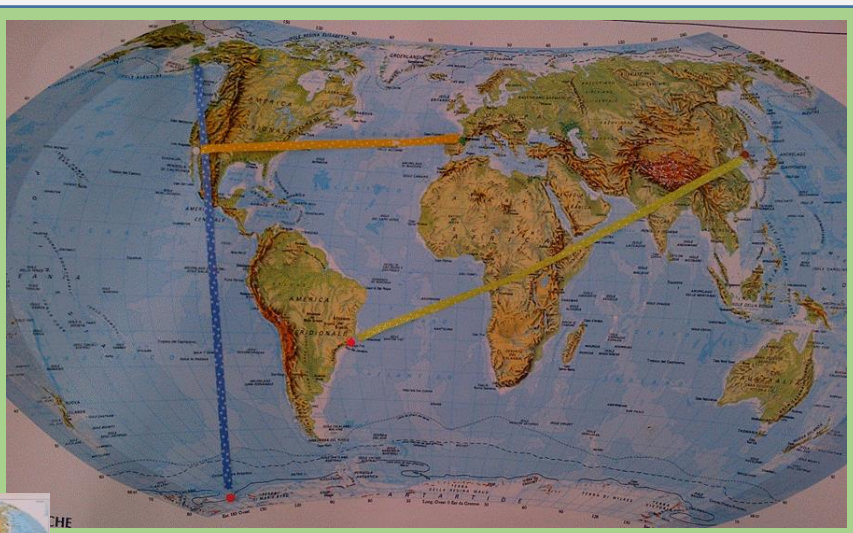
Differentiation strategies:

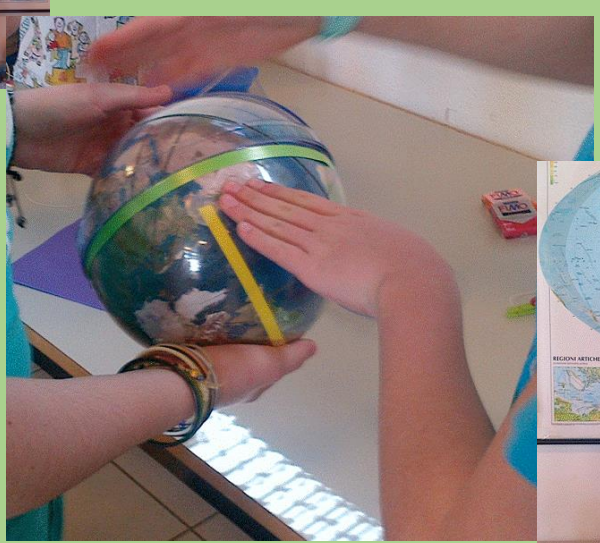
1. We have foreseen a two-speeds strategy. For the students very fast in “transferring lines of the sphere onto the 2D maps – and vice versa”, we proposed also “composite” counter-questions (e.g., if from London to San Francisco I had to stop in Miami, how would you change the route on the 2D map?). For the other, we require them “step by step” solutions (e.g. mark ten points on the sphere for the route London – San Francisco; report the ten points on the 2D map and interpolate them there).
2. A differentiating factor is also how skilled a student has become at finding whether a line on the sphere is curving or straight (requires using the spherical ruler or anyhow to apply analytical thinking – visual is not sufficient: the student first design the straight line between the extremities and then compare the initial line)

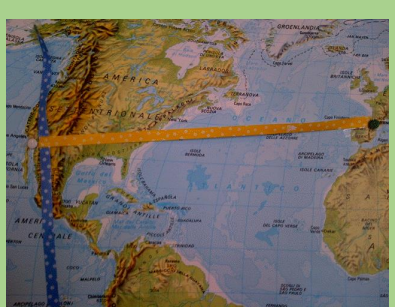
Group dynamics strategies:

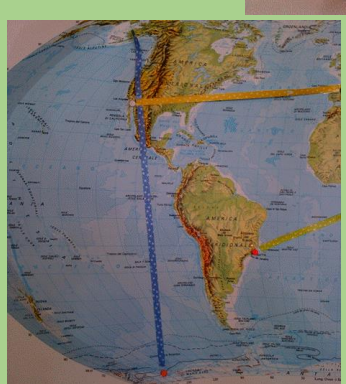
We have foreseen to let the students “keen on world’s destination” (two or three of them), to ask for specific routes or navigation questions

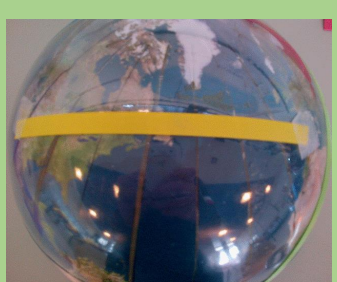
Picture of World Map with routes











Post-Implementation Observations

Straight paths on 2D maps: UD 3

UD 3– Lecture 6 (UD 3 Slides: 7 to 8 for all students) – Duration: 1 h

Socio-affective

1. What was said for the previous lectures – that the lectures "were not about mathematics", is particularly true for this lecture:
After having discovered about the deceiving appearance of lines of the worldmap,
Bernardo: We are doing geography, not mathematics … true?
2. Students were interested in discovering counterintuitive facts about the "travel geometry" and discussing about "exotic locations", but not to the extent we had foreseen. There was not a significant difference with the previous activities
3. The socio-empathic situations confirmed themselves as constant and good source of motivation, or interest – more than open questions on globe geography or Earth's shape.
4. In particular, we notice that the "quiz" formulation triggers their drive: quiz's, were they in the socio-empathic, or were they in the "long list questionnaire", immediately picked up 80% of the students full attention
5. For the major part of the students, the game is "to get it right"

Obstacles

1. Contrary to our expectations, the major part of the students had no difficulties in accepting, in front of the evidence of the Lénárt sphere, that straight lines of the 2D map were actually curving on the surface and vice versa.
Emilie: You see it immediately: there [she means, on the sphere] the land [she means the surface] is curved, while on the map [the surface underlying the line] is flat – so it is "appiattita" (flattened) [she means the original curving line]
Our interpretation is that their mental images of straight lines of charts as straight lines of the land have not had time to "ossify" – contrary to what happens to an adult that has been using local maps for years. Consequently, they may not have obstacles to assimilate the new notion. Indeed, the very few students who appeared to have used the world's maps – may be for personal interest in travel or as part of school geography, were those that showed more resistance to "assimilate" the concepts (actually, for them it becomes a question of accommodating the concepts)
Fiona: I have been to China [she shows me all cities she visited and travels on the Atlas] and we did not fly across the Artic [more or less the straight line] – the pilots surely know what is the shortest route!
2. Some students confused "being a straight line surface represented by the 2D map" with "being a straight line of the 2D-map-surface". We found this hard to be picked up:
Teacher, pointing to a line going from London to Los Angeles that on the 2D Atlas appear straight, after the student checked it on the Lénárt sphere being "curving"
Teacher: then, is the plane going straight or curving?
Elisa: ….. Straight
Teacher: … Straight?
Elisa: …. It's wrong, isn't it?
Teacher: I do not know, for I do not know how you got to it…
Teacher: Let me understand your thinking …
Elisa: to me it just looks straight …
Teacher: …straight
Elisa: you see [shows the straight movement with her hand]
Teacher: if I am following you, a micro-plane flying just above this map would be going straight … got it?
Elisa: that's what I say!
Teacher: that is correct, indeed
Teacher: yet, we are after a different question: imagine your micro-plane is flaying above the sphere [he shows different movements]
Teacher: in the case it follows this line on the sphere [the corresponding line on the sphere that was shown to deviate from a straight lien connecting London to L.A.) is this micro-plane going straight?
Elisa: like a bike, you mean? [she means a bike going straight]
Teacher: ya, like our mathematical bike
Elisa: no
Teacher: no..
Elisa: eh, it is curving
Teacher: so, that line, that looks straight on the map, was the trace of micro-plane going straight on the sphere?
Elisa: of course it is not! It is straight on the map only
Again, we have thought that this is possible if one is not accustomed to identify the 2D map with the Globe surface – as we are, and so we could not foresee such "free thinking and the consequent difficulty to understand our requests"

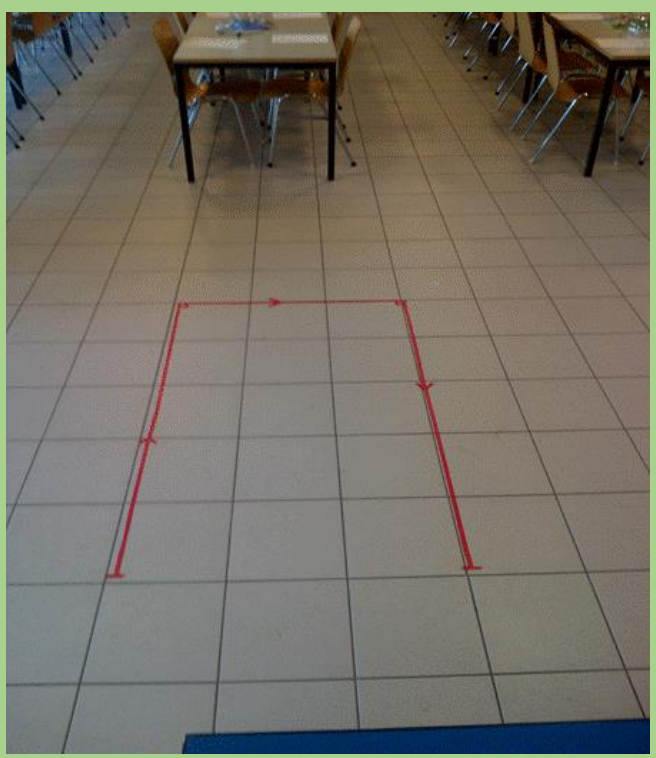
Non Euclidean behaviors of triangles: UD 4

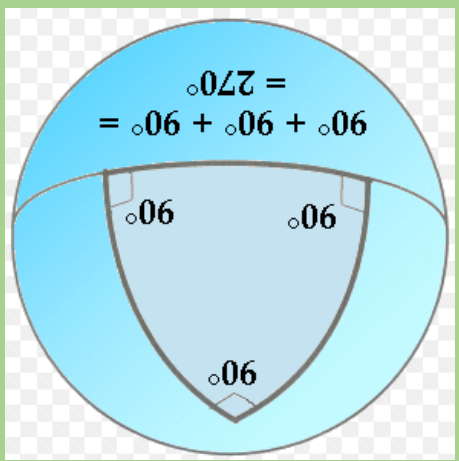
UD 4– Lecture 7 (UD4 Slides: 2 to 4 for all students) – Duration: 1.5 h

Premise
Differently from common approaches, our student comes to triangles only after many hours of work on critical prerequisites:
1. Straight lines on the sphere – he is used to consider, recognize and build straight lines and distinguish them from curving lines. This point is essential to avoid tracing triangles just by drawing "lines" – on the plan we do not have to think, straight lines come naturally from our hand drawing: not so on the sphere
2. Angles on the sphere – he has absorbed the image that every time we draw a couple of divergent straight radii leaving from a common origin, they were born flat and flat is the angle that is opening between them – for the surface is Euclidean locally. This point is essential to recognize the infinitesimal angles as "our angles" and not "curving angles obeying other rules"
3. Curvature – actually, any spherical angle, is not curved: its infinitesimal initial part keeps being flat: it is the underlying space that is curving: however we walk inside the angle, we will not see it curving. This point is essential to recognize the entire triangle as one of our triangle and not a "curved triangle obeying other rules"
These three points are essential to correctly draw spherical triangles and recognize them as our land triangles
Eventually, the discovery is that because straight lines never diverge, always come back to cross, one can "close" impossible "to be closed" triangles.
p. 1. The starting situation aims to puzzle the student and to involve him emphatically with the characters of the riddle.
Objective: To have the student
1. recognize that the spherical triangles and their angles are indeed his small flat triangles and flat angles "magnified"
2. discover that many triangles too wide to be closed on a small scale can be closed on a large scale
3. recognize that triangles on the spherical surface, on large scale, refute the "geometrical truth" about the sum of the angles
4. become aware that on large scale triangles may have degenerate shapes
Aimed outcomes: The student
1. can make examples of triangles exceeding the 180° value for the sum of the amplitudes of their internal angles
2. can discuss the equilateral and rectangle triangle
3. may discuss degenerate spherical triangles (overlapping vertices)
3. may discuss effects on our travel and movements on our land on large scale
4. may recall or produce paradoxical, empathic situations
Experiments and Materials:
Tape for drawing open triangles on the ground; Lénárt sphere, pencils, ruler and spherical protractor
Organization:
Groups of three with one Lénárt sphere and pencils in phase 1 (resolve the riddle)
Couples with full toolkit of Lénárt in phase 2: exploring and studying triangles on the spherical surface
Geometry concepts:
Angular excess of the sum of the internal angles of a triangle as characteristic of (positively) curved surface
Physical sciences concepts:
none
Geometry pedagogy/ cognition:
Learning centered on a problematic situation lived with empathy and leading to the discovery
Epistemology:
Impossible "to be closed" figures are closed by the curvature of the "underlying"
Obstacles and strategies:
Measuring angles. Angles appear "curved", the usual protractor becomes unreliable and the curvature of the surface makes more difficult to assess the amplitude of an angle "naked-eye". The strategy; just use the spherical protractor.
Puzzling changes in triangles shape. Triangles show different sorts of shapes on a spherical surface: the three vertices can be aligned or, even, two of them can coincide. The strategy: have the student work on triangles that maintain the vertices distinct and discuss the puzzling ones only with students that show interest and are ready to revise critically their definition of triangle

Differentiation strategies:
In phase 1 – advanced students are distributed in different groups, to make sure that every group arrives to a solution
In phase 2 – students "of the same speed" are put together to ensure that they can progress at their own pace or that they can start working on the advanced questions (cylinder, saddle, etc.)
Group dynamics strategies: baseline

The floor cannot close this triangle:
“the land is curved and make them bend to meet»





Post-Implementation Observations

Non Euclidean behaviors of triangles: UD 4

UD 4– Lecture 7 (UD4 Slides: 2 to 4 for all students) – Duration: 1.5 h

All students that would promptly know the property of the sum of the internal angles, when put in front of the equilateral triangle on the sphere with internal angles adding up to 270°, immediately replied that «it was wrong»
"It is wrong"
Anna: it's wrong: the sum must be 180°
Teacher: why?
Anna: there is no why: it is so
Teacher: then, I do you know it?
Anna: …. we cut the triangle [she means they had cut a flat paper triangle and recomposed a flat angle]
Teacher: could you have used a protractor to check it out?
Anna: of course!
Teacher: what does your protractor tells you of this spherical triangle?
Anna checks again: 90° + 90° + 90° … 270°
Teacher: 270°
Anna: it is right
Teacher: right?
Anna: it is so.
Teacher to the class
Teacher: we have [on the sphere] a triangle, with a side from which the other two sides come out with angles of 90°, right? (see picture previous page)
Marco and other: right
Teacher: here on the floor, I have the same thing: a side from which the other two sides come out the angles of 90° … the thing is that these sides cannot close in …they go off parallel. How is it that if we extend them very far out of the school land, they will eventually meet – as it is shown on the sphere?
Amelie: the land is curved and make them bend to meet

Yet, after the initial surprise, after 10 minutes, most of the students would not show any surprise at discovering many more triangles not respecting the normal behaviors .

We have proposed an interpretation schema for the similar situation with the circumference – schema that we propose for the triangles as well.

Degenerated triangles created confusions about "what to accept as a triangle"
Francesca: this is not a triangle [meaning the one with three vertices aligned on a straight line]
Teacher: why?
Francesca: one angle is missing…
Teacher: one angle…
Francesca shows the movement that brings the third point to be aligned to the other two, by which the vertex "disappear"
Teacher: how much was the angle of that vertex before disappearing?
Francesca: I do not know
Teacher: check it …
Francesca: 160° circa ..
Teacher: and when it disappears the angle is …
Francesca: …… 180°
Teacher: 180°
Francesca: no, I mean that the angle is 180° but the "tip" has disappeared [she means the vertex]
Teacher: the tip has disappeared …
Francesca: there is just a line there
Teacher: instead of ..
Francesca: the tip
Teacher: so triangles want tips …
Francesca: …. Yes, points
Teacher: and the point is not longer there …
Francesca: actually is there … but the tip is gone
Teacher: and the angle?
Francesca: it is there
Teacher: if we call triangle a figure with angles and points this is …
Francesca: a triangle
Teacher: if we want to have the tips …
Francesca: this is not a triangle!!

Non Euclidean behaviors: parallel lines: UD 5

UD 5– Lecture 8 (UD5 Slides: 2 to 3 for autonomous students; 4 to 5 for dependent students; 6 for all students) – Duration: 1.5 h

Premise
Epistemologically, this was the more difficult lecture – see the Box, Some of the epistemological challenges, and we had to leave out quite a lot.
For example: we could not help but starting by saying "that we have two parallel (straight) bikes and concluding by saying that there cannot be parallel straight lines". We were not satisfied by resolving this contradiction by the locally Euclidean character of the manifold. Nor were we satisfied with having to leave out connections (parallel transport) – without which we cannot even speak of two parallel bikes.
Indeed, the notion of parallelism is central to geometry. It is difficult to think of a geometry of the world that does not have parallelism! The Riemannian manifold value is also on this fundamental point: it does have it - locally! Of course, historically, we had to wait Levi Civita (Nozione di parallelism … 1917) to have a notion of parallelism on a Riemannian manifold, but he could do it, because Riemannian manifolds have local geometry.
On the other side, we have broadened the topic with respect to the traditional (global) view of non Euclidean geometry (for which "there are no straight parallel lines and no parts of straight lines that can be parallel"), by adopting a local view common in general relativity – by which the "change of the straight line separation" is the measure of the "curvature of space" (which is even variable!). Not only: we have made the common formulation (in bracket above), even weaker: "it makes sense to think of two bikes running side by side on straight lines for whatever distance - because the deviations are negligible on Earth's scale"
All students start from the same question about two friends start biking in parallel. Non advanced students have a track to follow to "gradually arrive to perceive the solution". After they receive an "explanation" of the resolution. Advanced students have to find their own "road to solution".
Objective: To have the student
1. recognize that it is mathematically impossible to run on large scale on lines that are both straight and parallel – either the straightness or the parallelism has to be given up
2. recognize that in a small region two lines may appear both straight and parallel, yet, on large scale they cannot keep being both
3. recognize that two "Earth's parallels" that are very near to the equator are practically parallel straight lines
4. has understood that through any point "sufficiently near" to a straight line, there can be drawn a parallel line that is as straight as desired
Aimed outcomes: The student
1. can show on the Lénárt sphere that trying to draw a parallel to a straight line leads to a secant
2. can show that any parallel line to the equator is an "Earth parallel" – a line which is not straight
3. can hint that a line that runs parallel to the equator with a separation of 1 m is practically straight
Experiments and Materials:
Lénárt sphere, pencils, ruler. If necessary, bikes and arrows ready outdoor.
Organization:
Phase 1: couples with one Lénárt sphere and toolkit, get either the advanced (called "sintetico") quiz or the "facilitated" (called "lungo")
Phase 2: ideas are shared and confirmed or discarded
Phase 3: in couples, students either rework the key points on the sphere or work on the advanced activities (cylinder)
Institutionalization:
By one summary slide + practical observations
Geometry concepts:
Change in parallel straight lines separation as an indicator of curvature of space.
There cannot be more than one ("geographical") parallel that is a straight line
Geometry pedagogy/ cognition:
It is critical to keep transferring the artifact's discoveries to the land by empathic and concrete situations – so that it will be fixed emotionally by the student.
Epistemology:
Parallel lines is "where all started" – and we could not include them. Yet, we believed the discoveries of the preceding lectures already very unsettling common images and common "language categorizations". Therefore, we did not emphasize the topic of this lecture.
Obstacles and strategies:
Epistemological-logico-linguistic: how can two small straight segments be parallel if there are no parallel straight lines?
Epistemological-kinesthetic: there are many possible; the three directly related to our laboratory are:
1. how is it that there are no straight parallel lines when if "I were" a car going straight I would leave two such lines?
2. how it comes that we can bike straight in parallel for kilometers without noticing anything and, in the end, we crash into each other?
3. how it comes that we crash always after the same distance – whether we are separate by 1 m or 100 km?
We did not define an ex-ante "obstacle management" strategy – for we were not sure about how students would have "seen" these obstacles. We decided to respond as required "on the field"
Differentiation strategies:
As mentioned, we differentiated with respect to "graduating the problem" – having observed a gradual separation of students in "problem solvers" and "exercise performers"
Group dynamics strategies:
baseline

Post-Implementation Observations

Non Euclidean behaviors: parallel lines: UD 5

UD 5– Lecture 8 (UD5 Slides: 2 to 3 for autonomous students; 4 to 5 for dependent students; 6 for all students) – Duration: 1.5 h

Socio-affective

By the time of this lecture, students have consolidated their attitudes towards this "way of doing mathematics":

1. Most of them were most of the time interested and willing to work without having to be pushed
2. Most of them were freely expressing their ideas – without the initial concern they had expressed for "being right or wrong"
3. Some students proposed ideas or explanations highly creative (but not appropriate) or were very quick to pick up the "essence" of the matter; these students would not correspond to any "profiling" as arising from common activities: one was very proficient; another quite shy and with variable results; another good in other disciplines, but not in mathematics

Cognition

1. All students could easily repeat and explain that there are no parallel straight liens on the sphere and on our land on large scale
2. Many students considered it already known, since the first work on the sphere and the discussion of the parallels
3. Almost all students expressed clearly that parallels are curving lines
4. Many students were clear that two parallel straight lines of our flat land will be bent by the curvature and will meet
5. Many students were clear that the two bikes will crash into each other
6. One or two students only could discuss if the two straight parallel lines are really parallel
7. Many students were able to make sense of local situation by the idea of "approximate or non precise observation", but t were able to explain the overall behavior as effect of curvature

Mathematical symbolism & proceduralim free

1. It seems that our students find common mathematics less interesting and more difficult (also) for the specific formalized semiotic codes; e.g., arithmetic, geometrico-literal – and the strict sequential process to be followed and coded in writing (for example: the sequence of operations to find the part of a whole given the fraction; or to find the height of a triangle given the relative side and the area)
2. it seems that students are more comfortable , at ease, with our problematic situations because they require comparatively more kinesthetic intelligence, visual processing and a logico-linguistic intelligence that, though very flexible (paradoxical logic), is very simple and concrete (indeed, we talk about curvature that you can touch – not things that require complex language constructs, e.g., "one third, is what you get if you divide in three and you get one)
3. It seems that students are less comfortable with procedural logic, symbols management, numerical and abstract operations
4. If the appearance has a factual basis, then we believe that it could confirm that their discomfort is due to stressing cognitive capabilities that are comparatively much less developed than the kinesthetic and general sensorimotorial
5. And if this was the case, then having built bottom up from sensorimotorial perspective our geometry would not be only a good solution to the problem of effective learning, but the only solution for students of their age: it is the only way in which they can process complex concepts
6. On the same line of thought, we could then, make the conjecture that by extending the sensorimotorial approach to other parts of geometry, and of mathematics as well, could benefit the students general well being and learning

Therefore, our lectures, being nearly "symbol" free and based on a heuristics, intuitions and deductions not strictly codified or organized -were perceived as non mathematics.

In addition, some students may have a comparatively higher kinesthetic intelligence than logico-numerical or procedural, and found more intuitive the geometry (topological and differential) concepts than the arithmetical ones - that appear to them non "logical" (that we interpreted meaning intuitive)

On a larger perspective, it seems to us, that may be major part of the students found themselves at relative ease, because the demand posed by the traditional lectures –much heavier in, is calling in intelligence or cognitive capabilities not so developed as those that we call in: kinesthetic, visual processing and using language in a very "loose" fashion

Teacher: Could you make me understand why you think it is not mathematics?
Gianni: eh, there are no numbers…
Teacher: also when do the constructions in geometry there are no numbers …
Gianni: yes, no, I want to say that … it is boring
Teacher: boring …
Gianni: the constructions – it is the same thing
Teacher: and you feel bored if you do the same thing – for example, playing the same videogame …
Gianni: … no, it is different
Teacher: how do you feel it different?
Gianni: I have to follow the rules…
Teacher: which rules?
Gianni: those of the constructions
Teacher: Ok, I understand, following rules is boring …
Gianni: that is it!
Teacher: and in our lectures …
Gianni: there are no boring rules …
Teacher: but there are difficult things … what do you say about "curvature"?
Gianni: it is not difficult for me [he shows with hands converging the idea of bending]
Teacher: a fraction is more difficult …
Gianni: yes, with rules and numbers …
[…]
Elisa: it is logical …
Teacher: [our mathematics] is logical …
Elisa: yes, of course if you go straight you will come back and we cannot go in parallel …
Teacher: and you feel better than in the usual lectures ..
Elisa: yes, I understand it …
Teacher: in the usual lectures you do not understand it ..
Elisa: no, yes, I understand but … it sin not logical…
Teacher: let me understand, I have a 4 apples and I take ¾ of them …
Elisa: … mmh… yes, ah yes, it is 3
Teacher: how you feel it?
Elisa: … no, yes, it is right, yes, but it is not logical
Teacher: even if we do a drawing [draws on the blackboard four balls and circles three] Elisa: I do not know: yes it is right but…

Post-Implementation Observations

Non Euclidean behaviors: parallel lines: UD 5 …. Parallels converge

Euclid’s flat surface: UD 6

NOT IMPLEMENTED

UD 6– Lecture 9 (UD 6 Slides) – Duration: 1 h

Objective: To have the student
1. recognize that by extending and magnifying beyond the horizon our personages the behaviors change radically – as seen
2. recognize that the behaviors do not change only if we 1. make mathematically flat the A4 sheet (not real) and 2. we magnify it (zoom out) in absolute void without end
3. recognize that this imaginative surface is that one on which school geometry is presented

Aimed outcomes: The student
1. is aware that both Euclid’s and spherical geometry can account for a flat surface – our experience of Earth surface is that of a flat surface
2. but spherical geometry hides an underlying curvature that strikingly changes the behaviors on large scale
3. whereas Euclid’s geometry is that one which looks the same however large is the magnifying glass or telescope
4. in Euclid’s geometry the sphere is just a figure as the cube and can be studied equally well
5. in the non Euclidean

Post-Implementation Observations

Euclid's flat surface: UD 6

UD 6– Lecture 9 (UD 6 Slides) – Duration: 1 h

NOT IMPLEMENTED

Attachment 3: Written tests and samples of students answers

Written tests and samples of students answers

In the following pages, we have included:

1. The original entry test
2. The original exit test
3. Samples of students answers for the entry test
4. Samples of students answers for the exit test

ENTRY TEST

Comportamento delle linee rette

Considera i punti P e Q:

1. Traccia tutte le linee rette che passano per tutt'e due i punti P e Q: quante sono?.....

2. Secondo te, è possibile trovare altri due punti in modo che di linee rette passanti per tutt'e due i punti ce ne siano non una sola, ma tante? E infinite?

   ……

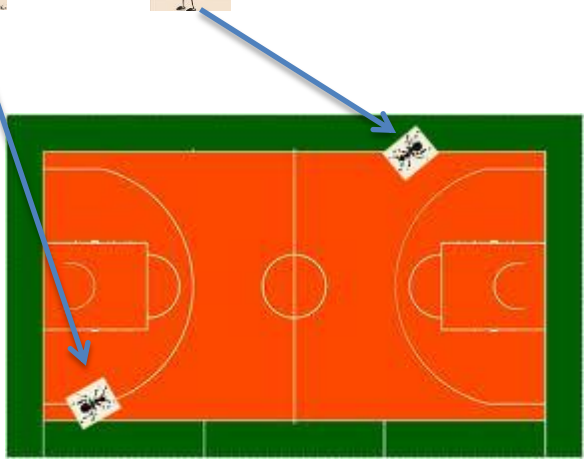
3. E se i due punti fossero distanti l'uno dall' altro migliaia di chilometri, la tua risposta precedente cambierebbe?

   ……

Se ti serve, questo spazio è per i tuoi schizzi e ragionamenti ☺

Moto rettilineo

Enza  e Pina  sono due ragazze su un bel campo da basket.



Enza si muove solo in bicicletta e va sempre e solo perfettamente dritto!

1. E' possibile che Enza arrivi da Pina facendo due percorsi diversi?

   …….

2. Se il campo da basket diventasse così grande da coprire tutta la Terra, e Enza fosse lontana migliaia di chilometri da Pina - la tua risposta precedente cambierebbe?

……..

……..

Se ti serve, questo spazio è per i tuoi schizzi e ragionamenti ☺

Un quiz

Carlos e Francesca vivono su un laghetto in un piccolo paese, vanno a scuola insieme e sono grandi amici.

Francesca parte in fretta e furia per una terra molto lontana - per il lavoro del papà - e non fa in tempo a salutare Carlos.

Dopo qualche tempo Carlos riceve un sms da Francesca, che dice:
“Scegli una direzione qualunque e immagina che puoi andare sempre dritto (attraverso le montagne e sopra i mari): alla fine arriverai sicuramente nella città in cui vivo!”

Carlos è molto confuso, ma si ricorda che Francesca è molto precisa. Tu cosa ne pensi?

……

……

……

Il nostro mondo

Ti trovi a Londra (London) e devi andare perfettamente dritto fino a Los Angeles – con un aereo.
L’unica condizione è che tu vada dritto.

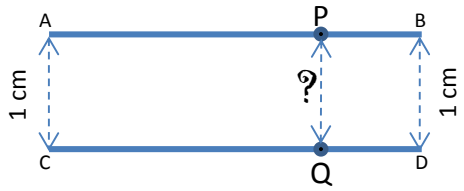
Disegna sulla cartina la traccia del tuo viaggio.

Comportamento delle linee rette: parallele

Considera i due segmenti paralleli qui sotto:

1. Immagina di poterli prolungare in linee rette senza fine e scegli la o le risposte giuste:
   Le rette che ottengo
   - ☐ Non s'incontreranno mai
   - ☐ S'incontrano solo da una parte
   - ☐ S'incontrano da tutte e due le parti
   - ☐ Non si può dire, dipende dalle rette
   - ☐ S'incontrano all'infinito
   - ☐ ……

Considera ora i due segmenti di linee rette AB e CD della figura qui sotto:



$\overline{AB} = \ell$ $\overline{CD} = \ell$

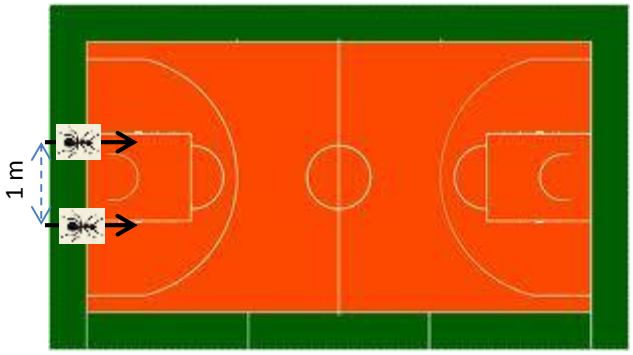
1. Scegli la o le risposte giuste:
   La distanza PQ:
   - ☐ Non può essere diversa da 1 cm
   - ☐ Puo' essere diversa da 1 cm
   - ☐ Puo' essere maggiore di 1 cm
   - ☐ Puo' essere minore di 1 cm
   - ☐ Puo' essere zero
   - ☐ ……

2. Se la lunghezza ℓ fosse migliaia di chilometri – la tua risposta precedente cambierebbe?
   Come? …….
   ……
   ……..

Se ti serve, questo spazio è per i tuoi schizzi e ragionamenti ☺

Moto rettilineo e parallele

Ora Enza e Pina sono tutt'e due in bicicletta



Partono una di fianco all'altra, separate da 1 m, vanno sempre dritte e alla stessa velocità.

1. Scegli la o le risposte giuste:
   - ☐ Continuano a rimanere separate da 1 m
   - ☐ Si avvicinano sempre più ma non si scontrano mai
   - ☐ Si avvicinano sempre più e poi si scontrano
   - ☐ Si allontanano sempre più
   - ☐ Ogni tanto si allontanano ogni tanto si avvicinano
   - ☐ Non si può dire
   - ☐ ……

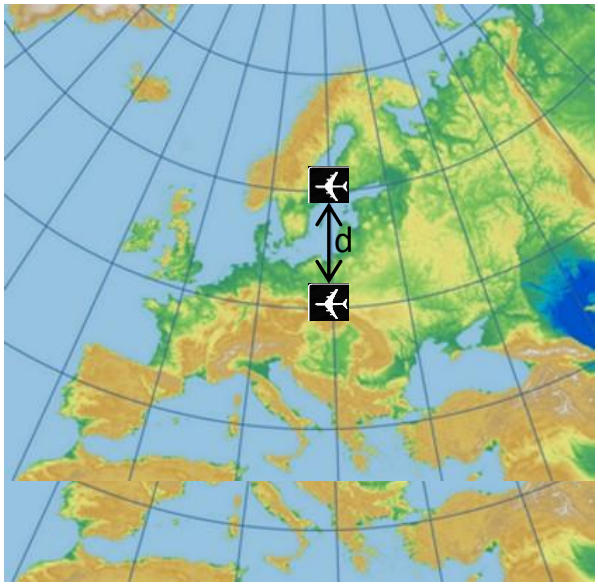
2. Se il campo da basket diventasse così grande da coprire tutta la Terra, la tua risposta precedente cambierebbe? Come?

……..

Il nostro mondo

Due aerei decollano dai punti indicati nella cartina della figura, si mettono paralleli ad una distanza d l'uno dall'altro e vanno perfettamente dritti (verso la sinistra della figura)

1. Disegna quelle che secondo te sono le rotte che seguiranno
2. La distanza d varia o rimane costante?.....



EXIT TEST

PRIMA PARTE

Rivedi il tuo test di ottobre, e scrivi sotto le risposte che vuoi cambiare. Se non vuoi cambiare o aggiungere nulla, lascia in bianco.

Comportamento delle linee rette (pag. 1)

1. ……..

2. ……..

3. ……..

Moto rettilineo (pag. 2)

1. ……..

2. ……..

Un quiz (pag. 3)

……..

……..

Il nostro mondo (pag. 3)

Perché è difficile rispondere giusto solo con la cartina?

……..

Cosa faresti per rispondere?

……..

Comportamento delle linee rette: parallele (pag. 4)

1. *crocetto la casellina: ……..*

1. *crocetto la casellina: ……..*

2. *crocetterei la casellina: ……..*

Perché? (puoi fare uno schizzo)……....

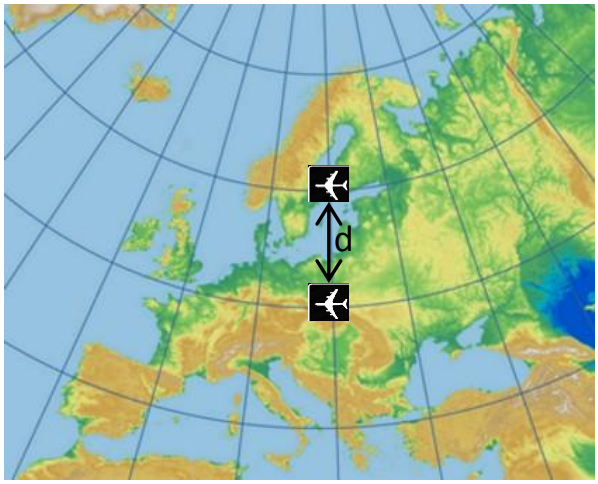
Moto rettilineo e parallele (pag. 5)

1. *crocetto la casellina: ……..*

2. ……

Il nostro mondo

Perché è difficile rispondere giusto solo con la cartina?
……...
Cosa faresti per rispondere?
……



SECONDA PARTE

Rispondi spontaneamente alle seguenti domande aiutandoti con schizzi o esempi a tuo piacere

Circonferenze e rette

Su un foglio una bici piccolissima gira intorno ad un punto O stando sempre alla stessa distanza da O.

1. La traccia che lascia è una …… di centro ……..

2. Può girare intorno ad O tenendo il manubrio dritto? ……

3. E se la superficie fosse grande come un campo da calcio? …….

4. E se la superficie fosse quella dell'intera Terra? ……..
Spiega (se vuoi anche con un disegno)

……..

……

5. Una circonferenza può essere una retta?
- No, mai; perché ……
- Non lo so; perché……
- Sì, perché conosco quest'esempio: ……..
- Mai, se la superficie è piatta
- Sì, se la superficie è sferica

6. Se aumenti il diametro di una circonferenza, è vero che la lunghezza della circonferenza aumenta?
- Sì, sempre; perché ……
- Non sono sicuro; perché ……
- No, perché
- so che: ……..
- Si, se la superficie è piatta
- No, se la superficie è sferica; perché succede che:……

E se tu fossi su una parte molto piccola di una superficie curva? ……

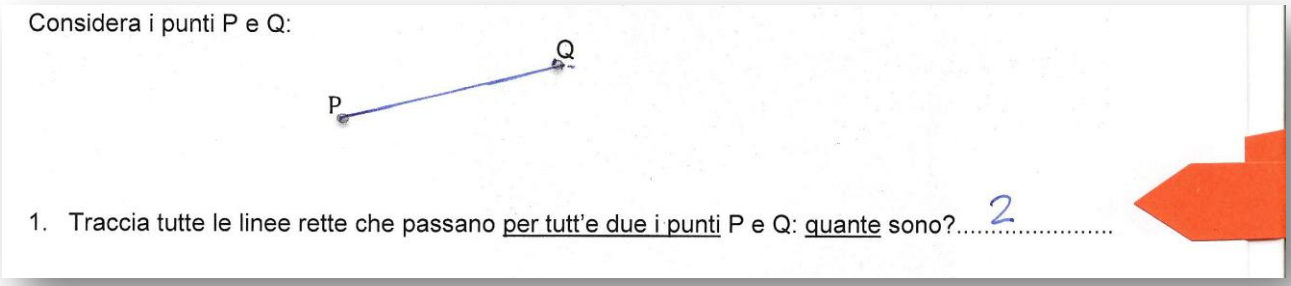
7. La somma delle ampiezza degli angoli interni di un triangolo è sempre 180°?
- Sì, sempre; perché ……
- Non sono sicuro; perché ……
- No, perché so che: ……..
- Si, se la superficie è piatta
- No, se la superficie è sferica; perché succede che:……

E se tu fossi su una parte molto piccola di una superficie curva? …….

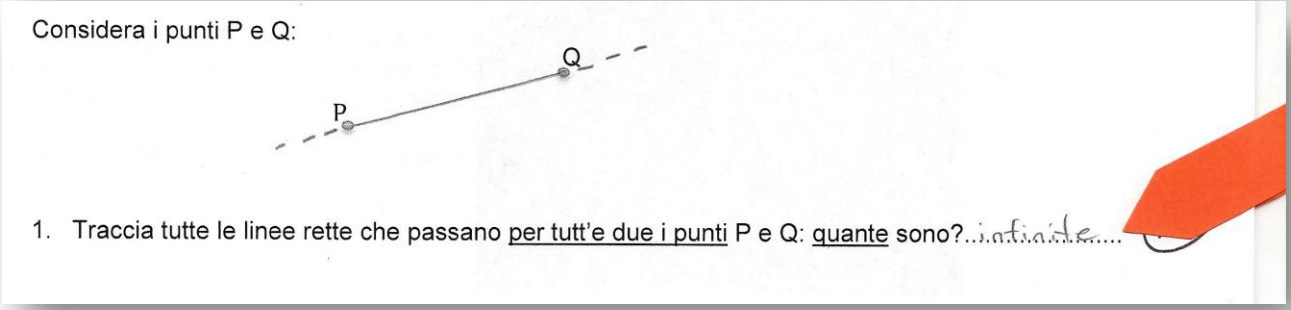
8. Per delle circonferenze grandi migliaia di chilometri e' vero che C/d è uguale a pi-greco? …..
Se la circonferenza è lunga 40'000 km quanto vale C/d? ……

ENTRY TEST: SAMPLE ANSWERS

Comportamento delle linee rette

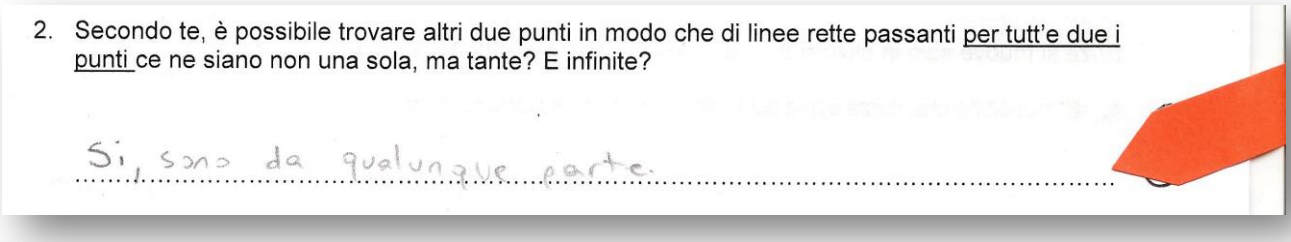

Considera i punti P e Q:

1. Traccia tutte le linee rette che passano per tutt'e due i punti P e Q: quante sono?.....2.....


Considera i punti P e Q:

1. Traccia tutte le linee rette che passano per tutt'e due i punti P e Q: quante sono?...infinite....

1. Secondo te, è possibile trovare altri due punti in modo che di linee rette passanti per tutt'e due i punti ce ne siano non una sola, ma tante? E infinite?

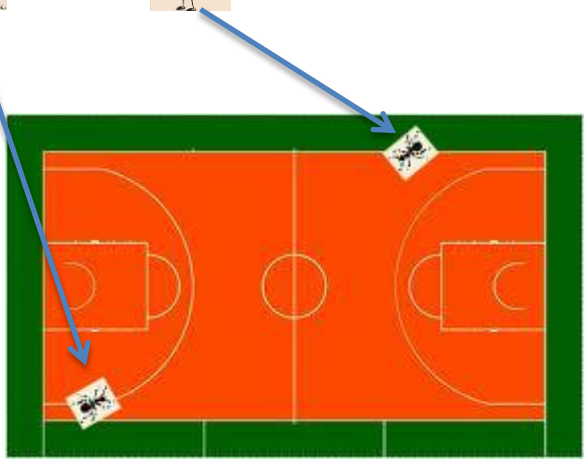

2. Secondo te, è possibile trovare altri due punti in modo che di linee rette passanti per tutt'e due i punti ce ne siano non una sola, ma tante? E infinite?

Si, sono da qualunque parte.


Si, perchè se si mettono altri due punti si possono mettere altre line

Moto rettilineo

Enza  e Pina  sono due ragazze su un bel campo da basket.



Enza si muove solo in bicicletta e va sempre e solo perfettamente dritto!

1. E' possibile che Enza arrivi da Pina facendo due percorsi diversi?

2. Se il campo da basket diventasse così grande da coprire tutta la Terra, e Enza fosse lontana migliaia di chilometri da Pina - la tua risposta precedente cambierebbe?

Sì perchè potrebbe fare il giro al contrario

Sì così enza fa marcia indietro e prima e oppoi arrivo a Pina

Sì, perché essendo il campo sferico enza può andare indietro per poi fare tutto il giro ed incontrare Pina

Un quiz

Carlos e Francesca vivono su un laghetto in un piccolo paese, vanno a scuola insieme e sono grandi amici.

Francesca parte in fretta e furia per una terra molto lontana - per il lavoro del papà - e non fa in tempo a salutare Carlos.

Dopo qualche tempo Carlos riceve un sms da Francesca, che dice:

"Scegli una direzione qualunque e immagina che puoi andare sempre dritto (attraverso le montagne e sopra i mari): alla fine arriverai sicuramente nella città in cui vivo!"

Carlos è molto confuso, ma si ricorda che Francesca è molto precisa. Tu Cosa ne pensi?

A senso perchè magari la terra è attorno al villaggio in qui abitano

Carlos è molto confuso, ma si ricorda che Francesca è molto precisa. Tu Cosa ne pensi?

io non ha nessuna idea

Secondo me non ha senza perchè se vai a destra e non a sinistra comunque dritto vai da un'altra parte

Può avere sensso soltanto se vivono sulla stessa linea (ad esempio l'equatore)

Carlos è molto confuso, ma si ricorda che Francesca è molto precisa. Tu Cosa ne pensi?

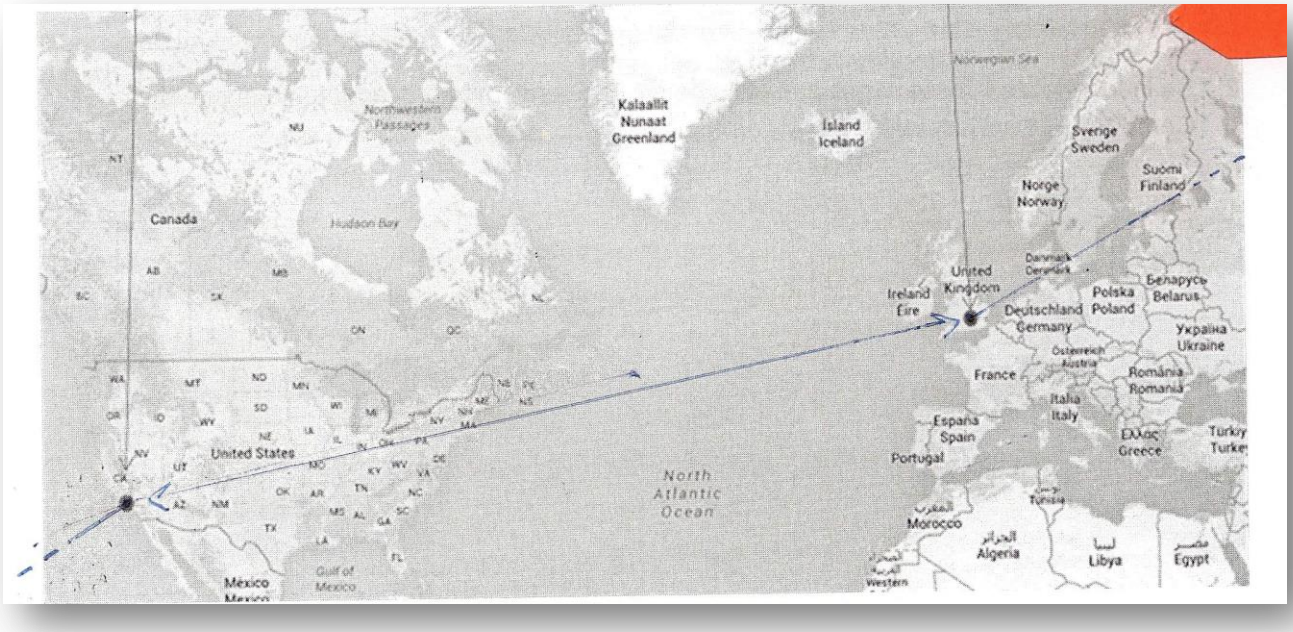
Non ha senso perché non c'è un punto in una sfera che si possa riassumere andando dritti in qualsiasi direzione.

Se condo me francesca ha ragione perchè continui a girare il mondo ma prima e appoi arrive dallàltra punto

Il nostro mondo

Ti trovi a Londra (London) e devi andare perfettamente dritto fino a Los Angeles – con un aereo.
L'unica condizione è che tu vada dritto.

Disegna sulla cartina la traccia del tuo viaggio.



EXIT TEST: SAMPLE ANSWERS

PRIMA PARTE

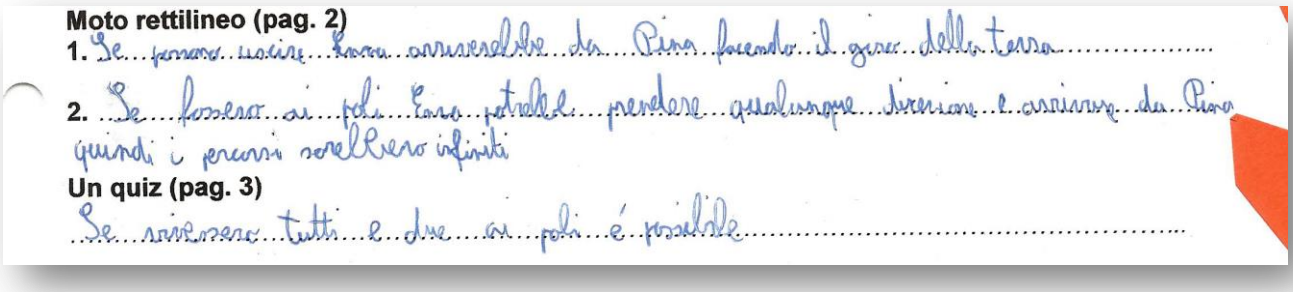
Rivedi il tuo test di ottobre, e scrivi sotto le risposte che vuoi cambiare. Se non vuoi cambiare o aggiungere nulla, lascia in bianco.

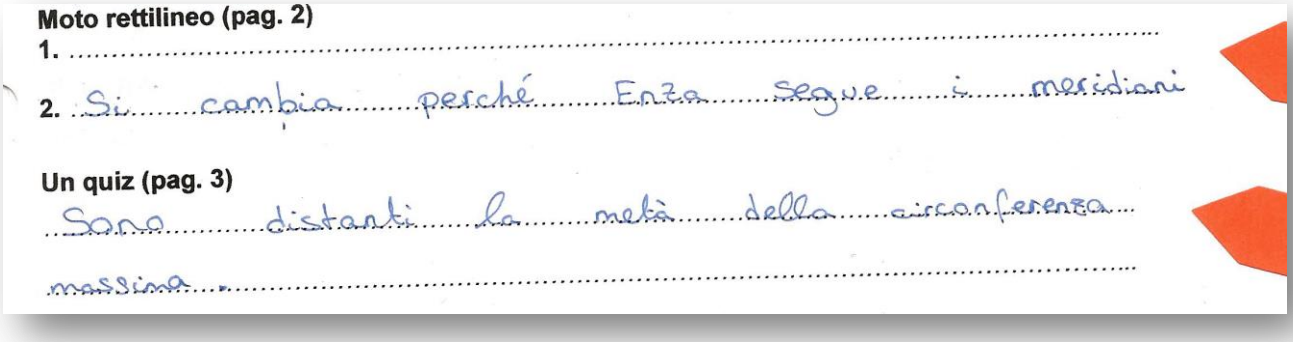

1. Una
2. No
3. Sì

Comportamento delle linee rette (pag. 1)


1.
2. Dovrebbe fare il giro della terra in modo diritto confronto all'altro punto e quindi sono 2 ma non infinite
3. Se fossero ai poli sono infinite le rette

Moto rettilineo (pag. 2)


Moto rettilineo (pag. 2)
1. Se potessero uscire Luna arriverebbe da Pisa facendo il giro della terra
2. Se fossero ai poli Luna potrebbe prendere qualunque direzione e arrivare da Pisa quindi i percorsi sarebbero infiniti
Un quiz (pag. 3)
Se arrivassero tutti e due ai poli è possibile


Moto rettilineo (pag. 2)
1.
2. Si cambia perché Enza segue i meridiani
Un quiz (pag. 3)
Sono distanti la metà della circonferenza massima

Moto rettilineo (pag. 2)

1. .....

2. Dipende, se è sul Polo Nord è scende và diritto

Un quiz (pag. 3)

Lei si trova dalla parte opposta del Mondo.

Un quiz (pag. 3)

Funzionerebbe solo se Francesca fosse ancora nel paese dove abita Carlos o fossero perfettamente opposti

Un quiz (pag. 3)

Francesca e Carlo abitano ai lati opposti della terra

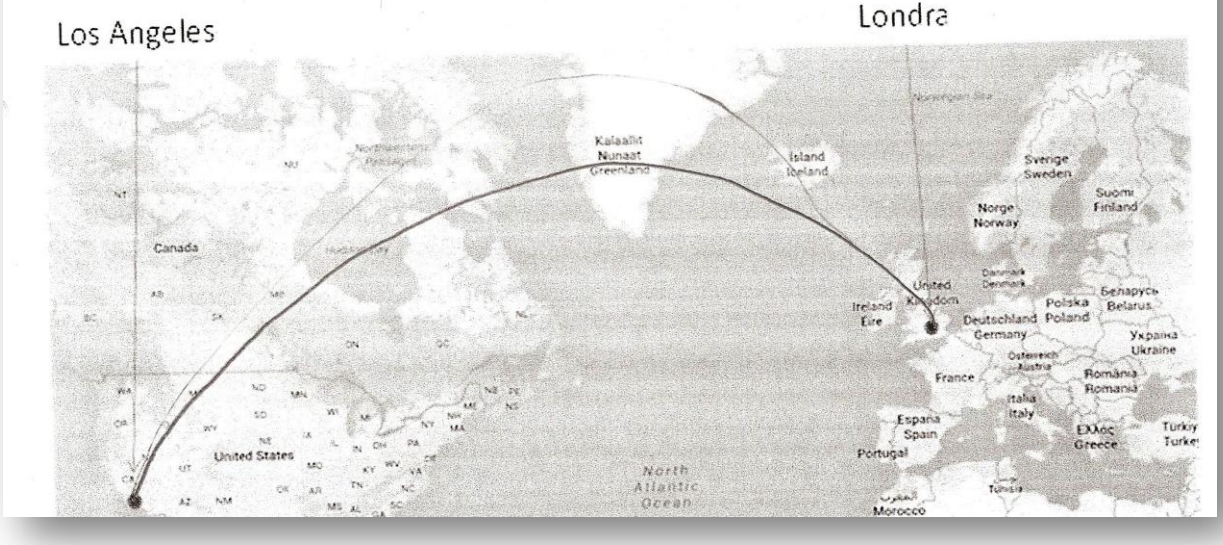
Il nostro mondo (pag. 3)

Perché è difficile rispondere giusto solo con la cartina?

Perché la cartina è piatta

Cosa faresti per rispondere?

Prenderei la sfera



Perché è difficile rispondere giusto solo con la cartina?

Perché le linee dritte sulla sfera diventano curve sulla carta

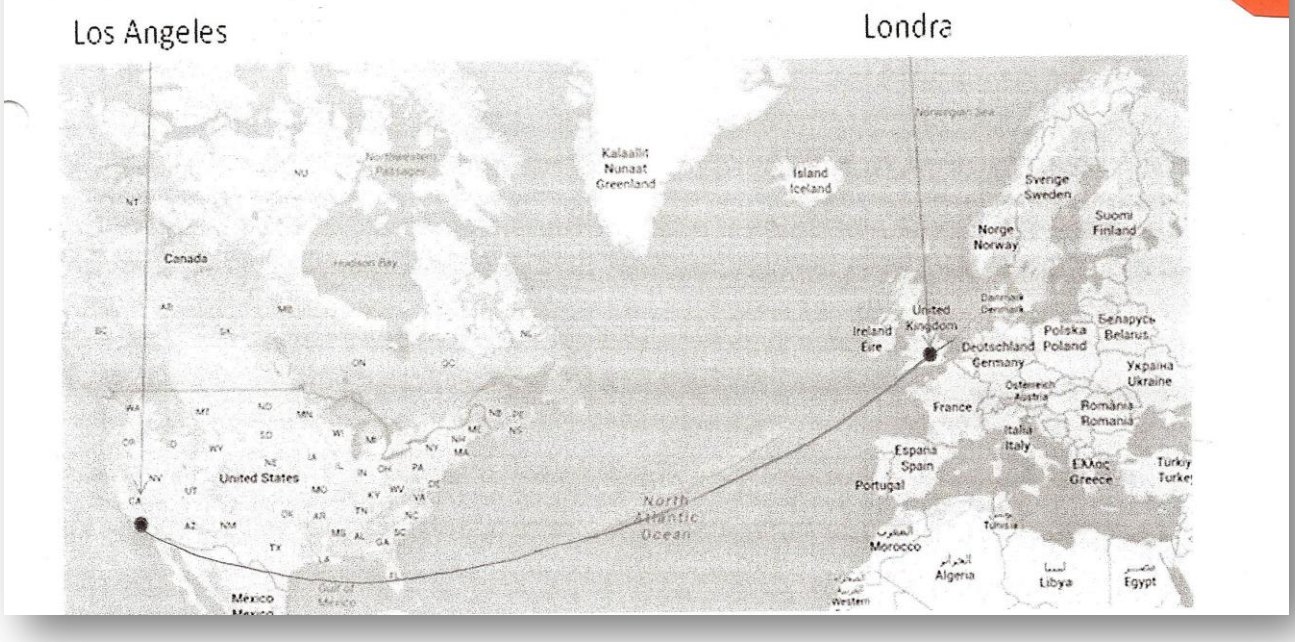
Cosa faresti per rispondere?

Perché è difficile rispondere giusto solo con la cartina?

* Perché la cartina è piana e le strade che sembrano più corte sono "false"

Cosa faresti per rispondere?

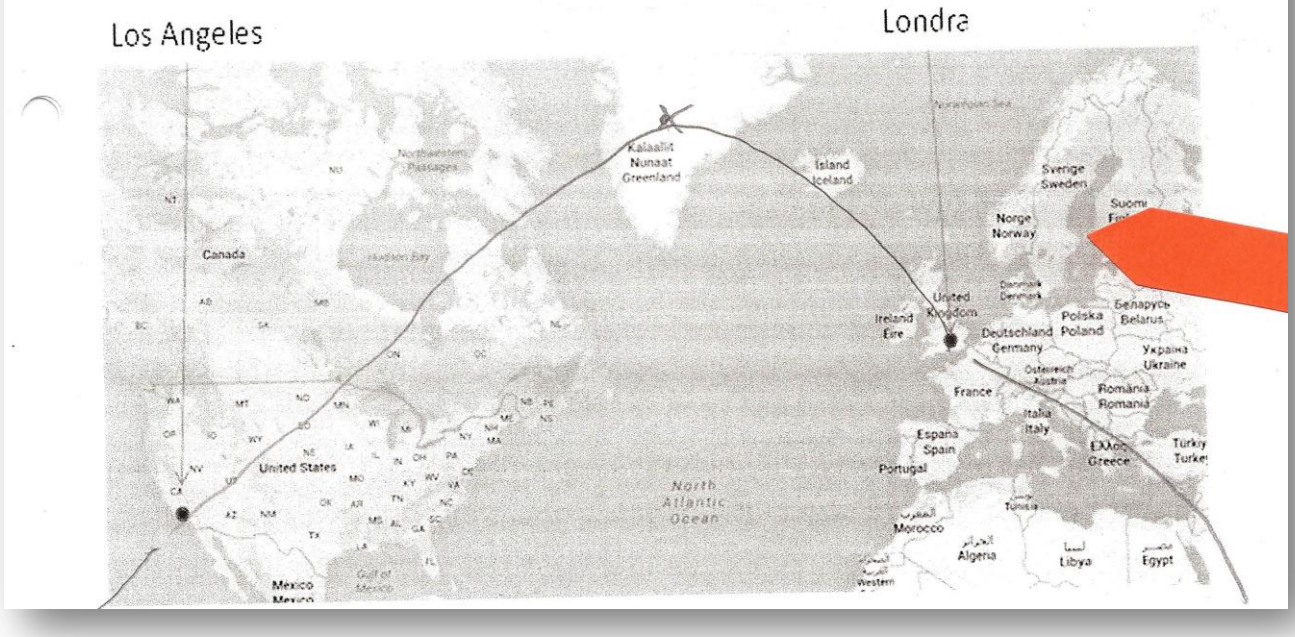
** Usere: un mappamondo (sfera)



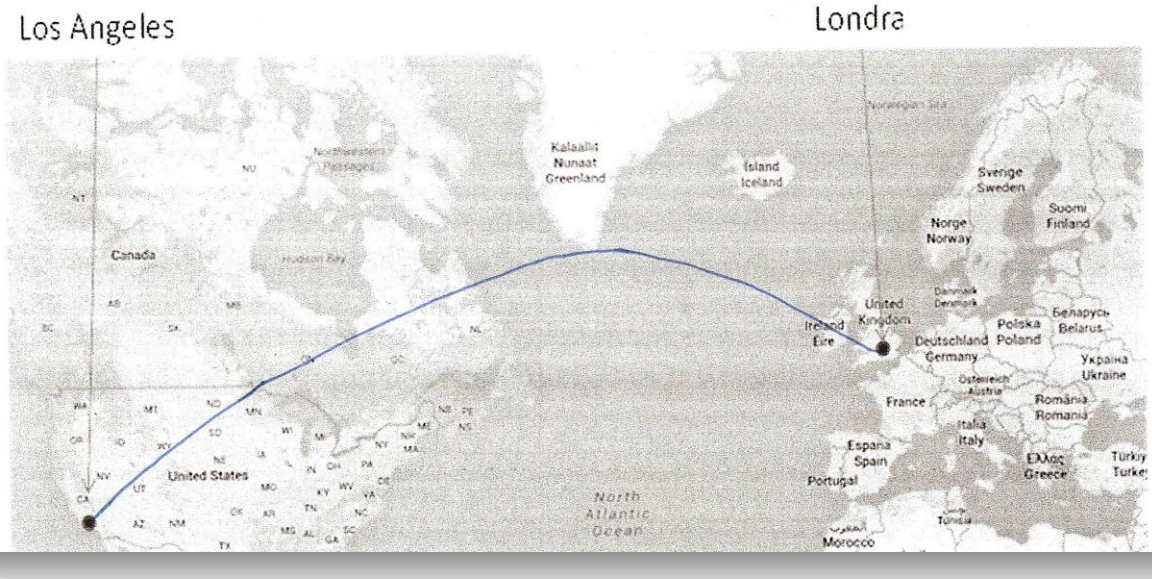
Perché è difficile rispondere giusto solo con la cartina?

Perchè la cartina è piatta invece il Mondo è sferico.

Cosa faresti per rispondere?



Perché è difficile rispondere giusto solo con la cartina?
Perché faremmo una linea dritta ma la linea dritta sulla carta non è dritta sulla sfera
Cosa faresti per rispondere?
Prenderei un mappamondo



Comportamento delle linee rette: parallele (pag. 4) e Moto rettilineo e parallele (pag. 5)

Comportamento delle linee rette: parallele (pag. 4)

1. *crocetto la casellina:* X

1. *crocetto la casellina:* ……..

2. *crocetterei la casellina:* ……..

Perché? (puoi fare uno schizzo)……..

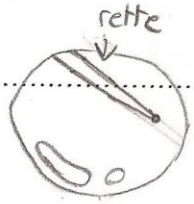
Comportamento delle linee rette: parallele (pag. 4)

1. *crocetto la casellina:* Si incontrano da tutte e due le parti.

1. *crocetto la casellina:* Non può essere diversa da 7cm

2. *crocetterei la casellina:* Può essere essere >, <, ≠ di 7cm

Perché? (puoi fare uno schizzo)……..



Moto rettilineo e parallele (pag. 5)

1. *crocetto la casellina:* ……..

2. Si avvicinano sempre di più e poi si scontrano

1. *crocetto la casellina:* 1.....

2. Si la prima a sinistra si incontreranno formando un triangolo

Il nostro mondo

Perché è difficile rispondere giusto solo con la cartina?

Perché non abbiamo trovato tutta la chiesa

Cosa faresti per rispondere?

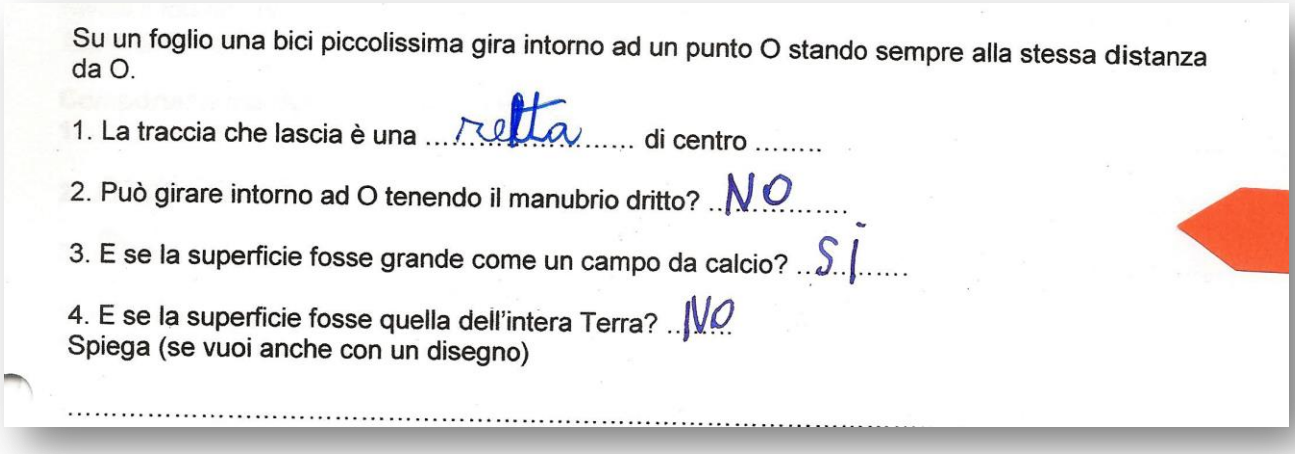
mi aiuterei con la chiesa

SECONDA PARTE

Rispondi spontaneamente alle seguenti domande aiutandoti con schizzi o esempi a tuo piacere

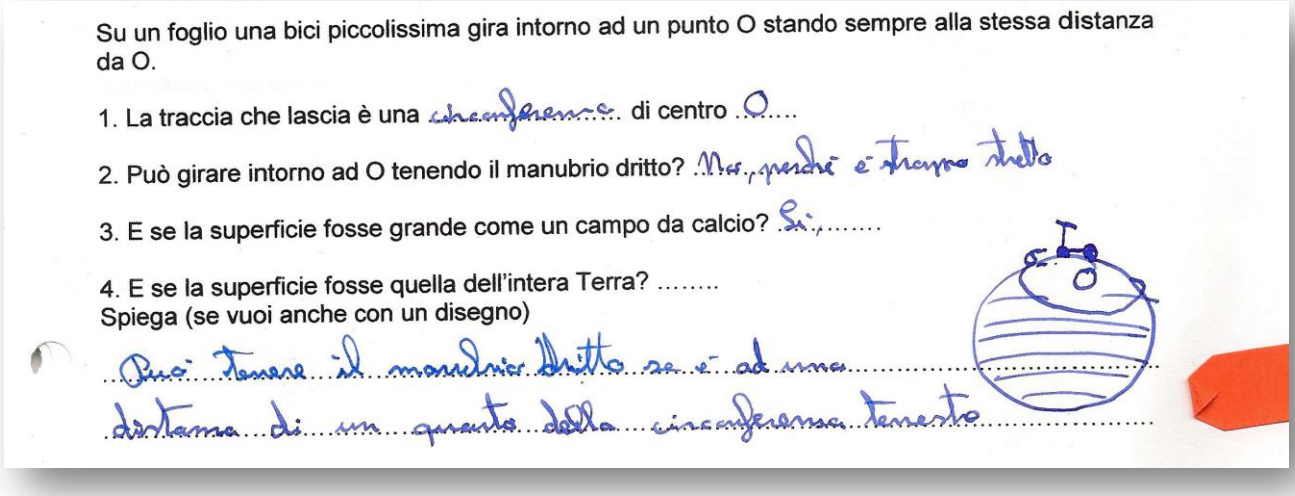
Circonferenze e rette

Su un foglio una bici piccolissima gira intorno ad un punto O stando sempre alla stessa distanza da O.


Su un foglio una bici piccolissima gira intorno ad un punto O stando sempre alla stessa distanza da O.

1. La traccia che lascia è una ...retta..... di centro .....
2. Può girare intorno ad O tenendo il manubrio dritto? ..NO.....
3. E se la superficie fosse grande come un campo da calcio? ..Sì.....
4. E se la superficie fosse quella dell'intera Terra? ..NO
Spiega (se vuoi anche con un disegno)

.....


Su un foglio una bici piccolissima gira intorno ad un punto O stando sempre alla stessa distanza da O.

1. La traccia che lascia è una circonferenza di centro ..O....
2. Può girare intorno ad O tenendo il manubrio dritto? ..No, perché è troppo stretto
3. E se la superficie fosse grande come un campo da calcio? ..Sì,.....
4. E se la superficie fosse quella dell'intera Terra? .....
Spiega (se vuoi anche con un disegno)

..Può tenere il manubrio dritto se è ad una distanza di un quanto della circonferenza tenerlo.....

1. La traccia che lascia è una circonferenza di centro .....

2. Può girare intorno ad O tenendo il manubrio dritto? NO

3. E se la superficie fosse grande come un campo da calcio? NO

4. E se la superficie fosse quella dell'intera Terra? SI
Spiega (se vuoi anche con un disegno)
Se andasse sull'equatore.

5. Una circonferenza può essere una retta?

- ☐ No, mai; perché .....
- ☐ Non lo so; perché.....
- ☒ Sì, perché conosco quest'esempio: sull'equatore
- ☐ Mai, se la superficie è piatta
- ☐ Sì, se la superficie è sferica

5. Una circonferenza può essere una retta?

- ☐ No, mai; perché .....
- ☐ Non lo so; perché.....
- ☐ Sì, perché conosco quest'esempio: .....
- ☐ Mai, se la superficie è piatta
- ☒ Sì, se la superficie è sferica

6. Se aumenti il diametro di una circonferenza, è vero che la lunghezza della circonferenza aumenta?

- ☐ Sì, sempre; perché .....
- ☐ Non sono sicuro; perché .....
- ☐ No, perché so che: .....
- ☒ Sì, se la superficie è piatta
- ☐ No, se la superficie è sferica; perché succede che:.....

E se tu fossi su una parte molto piccola di una superficie curva? No non cambia

- ☒ Sì, sempre; perché rimane costante
- ☐ Non sono sicuro; perché .....
- ☐ No, perché so che: .....
- ☐ Sì, se la superficie è piatta
- ☐ No, se la superficie è sferica; perché succede che:.....

7. La somma delle ampiezza degli angoli interni di un triangolo è sempre 180°?

- ☐ Sì, sempre; perché ……
- ☐ Non sono sicuro; perché ……
- ☐ No, perché so che: ……
- ☒ Si, se la superficie è piatta
- ☒ No, se la superficie è sferica; perché succede che: che il triangolo aumenta di più di 180

E se tu fossi su una parte molto piccola di una superficie curva? rimarra 1800

7. La somma delle ampiezza degli angoli interni di un triangolo è sempre 180°?

- ☐ Sì, sempre; perché ……
- ☐ Non sono sicuro; perché ……
- ☐ No, perché so che: ……
- ☒ Si, se la superficie è piatta
- ☐ No, se la superficie è sferica; perché succede che: ……

E se tu fossi su una parte molto piccola di una superficie curva? Immagina

8. Per delle circonferenze grandi migliaia di chilometri e' vero che C/d è uguale a pi-greco? …..
Se la circonferenza è lunga 40'000 km quanto vale C/d? ……

8. Per delle circonferenze grandi migliaia di chilometri e' vero che C/d è uguale a pi-greco? Si
Se la circonferenza è lunga 40'000 km quanto vale C/d? 2

8. Per delle circonferenze grandi migliaia di chilometri e' vero che C/d è uguale a pi-greco? NO
Se la circonferenza è lunga 40'000 km quanto vale C/d? 2

Attachment 4: Some epistemological challenges and exclusions

Some epistemological challenges and exclusions

Of the many challenges we met in the epistemological analysis, we provide a short and very informal account of those that, we think, may be worth sharing.

Contents

Challenges

1. Mathematical concept of curvature: Infinitesimals

We discuss about some mental images part of the process of grasping the idea of curvature – for a smooth surface in ordinary physical 3D space – assumed continuous at any order of magnitude. When trying to understand physically the idea of curvature - with the objective to optimize our ways to transpose it, we have immediately faced the intuitive, but logically ever puzzling, concepts of infinitesimals, e.g., actual entities resulting or not from a limit shrinking process and tiling of the surface (manifold), or in the form of affine vectors shrinking or not from a limiting process. We have found these epistemological and cognitive obstacles unsurmountable. Indeed,

1. how can the curvature of a tiny piece of surface, or line even, be "zero" and yet become "non zero" when we take very many of these little pieces? …" it is zero and yet it is not zero: it is infinitesimal".

This is the typical paradox of infinitesimals that has been surfacing in physical mathematics since at least Zenone's times.

From a psycho-cognitivist point of view, in our opinion, we do not see anything nonsensical about intuitive ideas of infinitesimals – or in the use of paradoxical language one may decide to do. This is already a good reason not to "ban" infinitesimals thinking from our students – a custom that we have observed very common. But there are good reasons also from the mathematical side.

The traditional "evasive strategies" of recurring to limiting processing - and dismissing the infinitesimals as non sensical, the "departed entities" of Berkley, are no longer the only mathematically rigorous solutions available to a large community - since at least the works of W. Noll and of A. Robinson, at the beginning of the 60's, who laid out infinitesimals based approaches to respectively continuum mechanics and mathematical analysis. Several mathematical systems have been developed since, and let us deal with infinitesimals (and infinites) in more rigorous ways (e.g., Robinson's NS analysis, hyperreal, surreal). Even more interesting, for our example, curvature of surfaces - infinitesimals notions in geometry are found in the works of some "builders" of modern differential geometry for relativity, e-.g., Ricci-Curbastro and Levi-Civita, whose "Calcolo Differenziale Assoluto" (Tensorial Calculus) is not only a central topic in mathematical physics, but gave to Einstein the mathematics to encode his ideas into a theory of gravitation. In this

field, as well there are many mathematical systems. We just mention the synthetic differential geometry (SDG) - in the words of Koch

"*It is a striking fact that differential calculus exists not only in analysis (based on the real numbers R), but also in algebraic geometry, where no limit processes are available. In algebraic geometry, one rather uses the idea of nilpotent elements in the "affine line" R; they act as infinitesimals. (Recall that an element x in a ring R is called nilpotent if xk = 0 for suitable non-negative integer k.) Synthetic differential geometry (SDG) is an axiomatic theory, based on such nilpotent infinitesimals. It can be proved, via topos theory, that the axiomatics covers both the differential-geometric notions of algebraic geometry and those of calculus*." (Koch, 2009, p. 1).

Our challenge was to trade off intuitive aspects of infinitesimal curvature and mathematical formalizations of infinitesimals.

2. Meaning of flat (and curved): empirically flat versus exactly or theoretically flat

We have found this point challenging, because of the public discussions about the geometry of space and universe foster confusion about the meaning of flat.

In common thinking and language, flat and curved are mutually exclusive properties: in a certain place, a surface is believed either to be curved or to be flat (not both), believed - not measured, not appearing. When mathematics and physics mix up – as in our case, things become more difficult to pin point. Theoretically is impossible detect an exactly zero curvature. Instead, we can theoretically detect any non zero curvature. This means that:

1. by direct measurement we could detect a Euclidean behavior –like zero geodesics separation acceleration, only if we had infinite precision (i.e., we can measure a real non rational number with all its digits). For the pragmatist philosophy we have taken on this issue, this amounts to say that is impossible

2. by direct measurement we can directly detect any non Euclidean behavior within our range of precision

3. by direct measurement we cannot directly detect any non Euclidean behavior out of our range of precision

In short: locally, our measures may tell us that the surface is curved, or they may tell us that it cannot be said (could be curved or flat). They cannot tell us that the surface is flat.

To stay clear from these issues we have generally used the expression “Empirically flat versus exactly or theoretically flat”

By empirically flat we simply mean that: “the surface is not distinguishable from a flat surface to the best approximation available”. More precisely, and for example, when the changes in the separation of parallel straight lines or deviations from 180° for the sum of the internal angles of a triangle can be explained as effects of finite precision of measurements. The surface may actually be curved – if the effects of the curvature are so small that they are not distinguishable from the effects of the errors of measurements: empirically flat does not imply that cannot be theoretically/ exactly curved.

To be exactly flat, we would need infinite precision and, as explained above, it is therefore impossible to prove, empirically, that a surface is flat, whereas it is true the contrary: any time in the future, a surface currently found empirically flat could turn out to be curved - as long as the precision of measurements increases.

For “curved”, it should be clear that is different: we can say that a surface is curved “without qualifying it further” – if deviations from Euclidean behaviors have been detected that cannot be offset by errors of measurements.

3. Elementary meaning of parallel lines

Discussing parallel lines on the spherical surface unearthed unexpected difficulties.

Traditional definition of straight parallel lines is of two straight lines with no intersection points – it is assumed that the straight lines are unbounded. Yet, there have been discussions on the definition since the times of Euclid (classified by Heath in three groups: Heath, 1956, Book 1, p.191).

With such definition, there is no discussion on a spherical surface: there are no parallel straight lines and no behaviors to investigate. And indeed this is what happens in many common non Euclidean laboratories.

Yet, in general relativity and Riemannian manifolds, the behavior of parallel straight lines is a central element in gauging geometry, e.g., non Euclidean aspects. The idea of parallelism is different and, as we discuss, very fruitful:

1. two unbounded straight lines for which the separation (we use separation, for we only need manifolds with affine connections, not a metric or distance) is constant.

In Euclidean geometry, the above statement is equivalent to saying that "two straight lines have the "same separation in two distinct couples of points". It is this statement that is more interesting for us: any couple of meridians for which we measure the separation in points symmetric with respect to the equator should be parallel. Yet, they are not in both senses: they intersect and their separation is not even constant.

In our laboratory, there is another type of parallelism that imposes itself at the psycho-cognition level of our students and that cannot be denied on the basis of traditional formal definitions: that of the geographical parallels (parallelism without straightness) – which can be informally seen in terms of "constant separation between two parallels and equal to the arch of any meridian crossing them".

But there is more: there can be parallelism and straightness on small scale – we can always draw two straight and parallel lines the separation of which is empirically (see previous discussion) constant (yet we know that if extended on large scale, they will eventually cross each other).

Another consequence of defining parallelism as constant separation is that a couple of curved geographical parallel lying on opposite side on an equator converge to a (common) straight line (the equator) as their separation converges to zero. In other words, train rails can be both straight and parallel – to a good degree of approximation, which can be increased to any level by shortening the separation.

We wish to bring attention on the fact that "assessing changes in separation" by a connection (affine connection) is simply the pre Weyl or pre Cartan parallel transport.

Our challenge was to understand some of the different micro problematic situations spawned by the above definition of parallelism and to decide how to deal with them in class.

4. Concept of "direction"

The concept of direction is ubiquitous in geometry and not just foundational to differential geometry and vector concepts. For our purposes, points we struggled with were:

1. in flat "space" (e.g., a surface) the concept of direction (or of a vector) that matters is that of "affine type": as the "arrow from point A to point B", which gives meaning to the "difference of

two points". This is good enough even for some non Euclidean flat spaces, e.g., special relativity Lorentz space-time, but breaks down when we move to "curved" space: the affine "rigid vector" goes off the (iper)-surface and by shrinking it through a limit process (as "when I asked a student to forget about his destination point and just tell me the very first first tiny little step, that is, "the" direction") we come to no other that the concept of tangent vector – which is eventually relative to one point (no longer two) and which is nothing else that the usual directional derivative ∂_u (we mean the operator, see Misner et al., 1973, p. 228). Being this what is lurking under our student difficulty to understand the concept of direction, it appears to us a reasonable conjecture that those that did not have the difficulty, did not have it not because they overcame this epistemological obstacle, but because they overlooked it and simply assimilated the idea superficially. Consequently, it appears to us that we could "risk" a non-orthodox hypothesis for this situation of mathematical difficulty: Fabio had difficulties, because his cognitive approach was deeper or more "epistemologically aware".

Indeed, when students worked on the Lénárt sphere, we asked them to use a "cloth spin" or a "toothpick" in equilibrium on the top point of the plastic sphere - from which they started moving. The "clothespin" or a "toothpick" stand for the tangent vector of the tangent space and mirror the arrow they had on the bike steer on the terrain.

5. Kinematical perspective

In modern differential geometry the kinematical sense of geometry is indissolubly intertwined with the static sense – as Poincare' more than hundred years ago was asking. The trouble is that our students do not even have a concept of velocity - in the best case, they have the concept of speed, in the worst, simply sensorimotorial based intuitive cognition. Mathematical physics concept of velocity on curved surfaces, presents even deeper epistemological challenges - regularly returning to infinitesimals-reasoning (see previous paragraph about curvature).

This issue is particularly thorny in the situation of the two bikes supposedly initially aligned (parallel) and travelling at the same speed on straight lines. If the bikes are not infinitesimal, our objectives and our talking become immediately self-contradictory: "the bikes lay parallel on two straight lines that we want to prove that are not parallel"! We could pretend to resolve the paradox by simply uttering it as "they are parallel and yet non parallel". Yet, at a more refined analysis, things are here richer. All of the infinitesimal vectors orthogonal to the equator are parallel – and

indeed, they are on the natural connection (parallel transport). However, all of them do lay on straight lines (meridians) that are not parallel. In a sense, some of them are “less parallel” than other: two “far away” vectors lie on meridians with a high rate of diminishing separation than two very near vectors. The synthetic rate is just the angle (longitude). It is only when both dimensions (bike and shift) are infinitesimal that there is flat parallelism.

As it is clear, parallelism, velocity, direction – are entangled challenges. We could not find an easy way out.

Exclusions

We report about some epistemologically key aspects that we had to exclude from our transposition.

1. Exclusion of the tangent bundle point of view

We have discussed manifolds with the historical approach, instead of "seeing" them as the abstract entity identified by the tangent bundle. Our choice is in line with the pedagogical approach used with the students – directly on the "manifold": locally, on the basketball court. For the same reason we have left out any discussion on redefinition of Riemannian manifolds in terms of fiber bundles.

2. Exclusion of non-Euclidean behaviors in flat space from the laboratories

We have mainly discussed non Euclidean spaces that can be discussed by "curved manifolds" – leaving out any discussion of non Euclidean spaces on flat manifolds. This exclusion has a scientific high-price, for, although on large scale the space time is a curved manifold, on our world small scale (locally) it is, empirically, a flat non Euclidean manifold (special relativity's Lorentzian manifold). We had to exclude it, for a transposition of this type of non Euclidean character, it seems to us, would have required to bring in metric (distance geometry) – an approach to geometry that we endeavor to avoid because of the profile of our class – they had yet to be introduced to the elementary Cartesian grid (Z instead of R). Had we had a class with curricular differentiation, we would have certainly explored a transposition strategy.

3. Exclusion of explicit discussion of gravity force

By having students going around the Earth (with their imagination and on the artifacts) it is inevitable to face the issues of "things falling off" and of "the force of gravity". We were committed not to talk and avoid as much as possible discussions about the force of gravity – not just because the force of gravity is a well-studied obstacle for middle school students, but especially because the very concept of "force" is at odds with the whole epistemology behind this work (general relativity). Indeed, the central idea of this course is that of "curvature" and, in our opinion, the most striking example of its significance is how it swept away the force of gravity: since Einstein there has not been any need for such a thing as a "force of gravity": nearly 100 years ago,

Albert Einstein freed science from the idea of a force acting at distance (in no time) postulated by Newton (to account for the movement of a falling apple as well as of the Moon) and recognized instead the curvature of space-time as ruling the movement of mass (and mass as ruling the curvature of space time)[1]. Yet, contrary to widespread misconceptions, this result does not require general relativity – it is already available to Newton's gravitation: its mathematical theory of gravitation can be rewritten without using the "force" and using instead the curvature of space time – so was shown by Cartan (as we discussed in our epistemological analysis). Therefore, had we to include gravity, for example in a higher middle school class, we would have considered exploring the possibility to prepare a lab on Newtonian gravity & curvature of space time, not force.

4. Exclusion of string theory

In our analysis, we decided to avoid any reference to n-dimensional spaces of string theory (n > 5), because we are far from understanding its epistemology. In addition, a number of mathematical physicist look at it as a theory "Not Even Wrong" – a way used by Pauli to point to theories that are so incomplete that cannot even make predictions by which could be tested (Woit, 2006).

[1] This is the Einstein's field's equations